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TECHNICAL REPORT NO. 1036

March 2001

ACCURATE LOWER TOLERANCE LIMITS FOR THE NORMAL RANDOM EFFECTS MODEL

by

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Supported in part by the Army Research Office under Grant Number DAAD19-99-1-0185.

1 Introduction and Summary

In this paper we obtain lower tolerance limits for the balanced normal random effects model. Government certification and qualification requirements as well as contract specifications are frequently stated in terms of lower tolerance limits. Thus, the efficient determination of such tolerance limits is important in technological application of statistics.

This study was motivated by the need for qualification of polymer composite for aircraft construction. The fabricator prepares material in rolls, each of which are several hundred feet in length (when unrolled). The various rolls correspond to the batches in the prior discussion.

Thus let

$$X_{ij} = \mu + b_i + e_{ij}, \quad i = 1, 2, \ldots, I, \quad j = 1, 2, \ldots, J \tag{1}$$

where X_{ij} is the jth observation from the ith batch. The b_i's and e_{ij}'s are mutually independent normally distributed random variables with $E(b_i) = E(e_{ij}) = 0$ and variances σ_b^2 and σ_w^2 respectively.

Let

$$\hat{\mu} = \sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij}/IJ, \tag{2}$$

$$\bar{X}_{i\cdot} = \sum_{j=1}^{J} X_{ij}/J, \tag{3}$$

$$SS_B = \sum_{i=1}^{I} J(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2, \tag{4}$$

$$SS_w = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2. \tag{5}$$

Further, define

$$s_B^2 = SS_B/(I-1), \tag{6}$$

and

$$s_w^2 = SS_w/I(J-1). \tag{7}$$

Let P and γ, $0 < P, \gamma < 1$ be given. Then $\hat{u}(P,\gamma)$ is a $100\gamma\%$ lower tolerance level for the $100(1-P)^{th}$ percentile, u_p, of the random variable X if

$$P\{\hat{u}(P,\gamma) \leq u_p\} \geq \gamma. \tag{8}$$

Clearly, if X is a normally distributed random variable with mean μ and variance σ^2, (8) may be written

$$P_{\bar{X},s}\{\bar{X} - ks \leq \mu - K_p\sigma\} \geq \gamma; \tag{9}$$

where $K_p = 100P^{th}$ percentile of the standard normal distribution, $k = k(n,\gamma,P)$ is called the tolerance limit factor; $\bar{X}$ and s are the sample mean and sample standard deviation of a normally distributed random sample with mean μ and variance σ^2.

For the normal random effects model (1),

$$E(X_{ij}) = \mu \tag{10}$$

$$\sigma_{X_{ij}}^2 = \sigma_x^2 = \sigma_b^2 + \sigma_w^2 \tag{11}$$

$$E(s_B^2) = \sigma_B^2 = J\sigma_b^2 + \sigma_w^2 \tag{12}$$

$$E(s_w^2) = \sigma_w^2. \tag{13}$$

Procedures for obtaining lower tolerance limits for normal random effects models have previously been given by Lemon [1977] and Mee and Owen [1983].

Lemon proposed the tolerance limit

$$\hat{\mu} - k_L s_x, \tag{14}$$

where

$$s_x^2 = s_B^2/J + (1-1/J)s_w^2 \tag{15}$$

and

$$k_L = t_{I-1}(\delta, \gamma) s_B / \sqrt{IJ} s_x; \tag{16}$$

$t_{I-1}(\delta, \gamma)$ is the $100\gamma^{th}$ percentile of a noncentral t distribution with $I-1$ degrees of freedom and noncentrality parameter δ, and

$$\delta = \sqrt{IJ} K_p \frac{\sigma_x}{\sigma_B} = \sqrt{IJ} K_p [(R+1)/(JR+1)]^{1/2}, \tag{17}$$

where $R = \sigma_b^2/\sigma_w^2$. Since the tolerance limit factor k_L depends on R, Lemon proposed replacing R by $\hat{R} = s_b^2/s_w^2$ in (17), where $s_b^2 = (s_B^2 - s_w^2)/J$. Lemon's procedure gives extremely conservative tolerance limits since his estimator of variability is based on s_B^2 which has $I-1$ degrees of freedom.

Mee and Owen [1983] proposed the tolerance limit

$$\hat{\mu} - \frac{t_f(\delta, \gamma)\ s_x}{\sqrt{IJR^*}}, \tag{18}$$

where $t_f(\delta, \gamma)$ is the $100\gamma^{th}$ percentile of the noncentral t distribution with

$$f = \frac{(R^*+1)^2}{(R^*+1/J)^2/(I-1) + (1-1/J)/IJ} \tag{19}$$

degrees of freedom and noncentrality parameter $\delta = \sqrt{IJ} K_p \sigma_x / \sigma_B$,

$$R^* = \max(0, (F_\eta Z - 1)/J); \tag{20}$$

(19) is obtained from the Satterthwaite approximation and F_η is the $100\gamma^{th}$ percentile of the F distribution with $I(J-1)$ and $I-1$ degrees of freedom and $Z = s_B^2/s_w^2$.

The tolerance limit given by (18) is also conservative (but less conservative than (14)), Since R^* is basically an upper confidence limit for R, R is intentionally overestimated. Also η is set to .85, which is approximately correct when $R \to \infty$ but is much too high for low values of R.

This problem and related problems have also been considered by others. Specifically, Bhaumik and Kulkarni [1991] discussed obtaining one-sided tolerance limits for the unbalanced one-way ANOVA random effects model. If the variance ratio R is known, they claimed improvements over the Mee-Owen procedure for large values of R. Vangel [1992] treated one-sided tolerance limits for one-way balanced random effects models using methods based on techniques developed by Welch [1947] and subsequently Trickett and Welch [1954] and Aspin [1948] for dealing with the Behrens-Fisher problem. Bagui, Bhaumik and Parnes [1996] derived tolerance limits for unbalanced m-way random effects ANOVA models based on estimation of the variance components. Also Beckman and Tietjen [1989] generated two-sided tolerance limits for the balanced random effects ANOVA model. They provided extensive tables of tolerance limit factors.

The purpose of the present method is to obtain tolerance limits with actual coverage virtually equal to the nominal coverage γ.

2 A Tolerance Limit for the Normal Random Effects Model

It is customary to describe the balanced normal random effects model by means of the ANOVA table given below.

Table 1. One-Way Analysis of Variance

Source	df	SS	MS	EMS
Between	$I-1$	SS_B	s_B^2	$J\sigma_b^2+\sigma_w^2$
Within	$I(J-1)$	SS_w	s_w^2	σ_w^2
Total	$IJ-1$	SS_t	s_t^2	

Let α, β, γ be arbitrary positive constants and let U and V be independent chi-square random variables with m and n degrees of freedom respectively.

Let $Z = \gamma U/V$ and $W = \alpha U + \beta V$. Also $Z = \dfrac{\gamma m}{n} F_{m,n}$, where $F_{m,n}$ is a random variable with the F-distribution with m and n degrees of freedom. It is easy to see that the conditional probability density function of W given $Z = z$ is

$$f_{W|Z}(w) = \frac{\left(\frac{\gamma+z}{\alpha z+\beta\gamma}\right)^{(m+n)/2} w^{(m+n)/2-1}}{\Gamma(\frac{m+n}{2})\ 2^{(m+n)/2}} \ \exp\left[\frac{-(\gamma+z)w}{2(\alpha z+\beta\gamma)}\right]. \tag{21}$$

Let

$$T = \frac{(\gamma+Z)W}{\alpha Z+\beta\gamma}. \tag{22}$$

Then T and Z are independent and T is a chi-square random variable with $m+n$ degrees of freedom.

If we now set $U = (I-1)s_B^2/\sigma_B^2$ and $V = I(J-1)s_w^2/\sigma_w^2$, then

$$T = \left[(I-1)\frac{s_B^2}{s_w^2} + I(J-1)\frac{\sigma_B^2}{\sigma_w^2}\right]\frac{s_w^2}{\sigma_B^2} \tag{23}$$

has the chi-square distribution with $(I-1)+I(J-1) = IJ-1$ degrees of freedom.

Let u_p denote the $100(1-P)^{th}$ percentile of the normal population X with mean μ and variance σ_x^2, K_p the $100P^{th}$ percentile of the standard normal distribution, and let $\hat{\mu} - k\sqrt{T}$ be a $100\gamma\%$ tolerance limit for u_p; then $u_p = \mu - K_p\sigma_x$ and

$$P_{\hat{\mu},T}\{\hat{\mu} - k\sqrt{T} \le \mu - K_p\sigma_x\} \ge \gamma. \tag{24}$$

Define $Y = \sqrt{IJ}(\hat{\mu}-\mu)/\sigma_B$ and

$$\delta = \sqrt{IJ}K_p\sigma_x/\sigma_B, \tag{25}$$

then k must satisfy

$$P_{Y,T}\left\{\frac{Y+\delta}{\sqrt{T/(IJ-1)}} \le \sqrt{IJ(IJ-1)}k/\sigma_B\right\} \ge \gamma. \tag{26}$$

Since Y has the standard normal distribution and T has the chi-square distribution with $IJ-1$ degrees of freedom, the random variable $(Y+\delta)/\sqrt{T/(IJ-1)}$ has the noncentral t distribution with $IJ-1$ degrees of freedom and noncentrality parameter δ. Hence, we have

$$k = \frac{t_{IJ-1}(\delta, \gamma)\sigma_B}{\sqrt{IJ(IJ-1)}}, \tag{27}$$

where $t_{IJ-1}(\delta, \gamma)$ denotes the $100\gamma^{th}$ percentile of the noncentral t distribution with $IJ-1$ degrees of freedom and noncentrality parameter δ. Let τ denote the ratio of the expected mean squares (EMS), that is

$$\tau = \frac{\sigma_B^2}{\sigma_w^2} = JR + 1. \tag{28}$$

Set

$$\hat{\mu} - k\sqrt{T} = \hat{\mu} - k_T \cdot s_w, \tag{29}$$

where

$$k_T = \frac{t_{IJ-1}(\delta, \gamma)}{\sqrt{IJ(IJ-1)}}\sqrt{(I-1)\frac{s_B^2}{s_w^2} + I(J-1)\tau}. \tag{30}$$

Therefore, the tolerance limit can be written as

$$\hat{\mu} - \frac{t_{IJ-1}(\delta, \gamma)}{\sqrt{IJ(IJ-1)}}\sqrt{(I-1)\frac{s_B^2}{s_w^2} + I(J-1)\tau} \cdot s_w. \tag{31}$$

Referring to the noncentrality parameter δ in (25), let

$$\frac{\sigma_x}{\sigma_B} = \left[\frac{1}{J}\left(1 + \frac{J-1}{\tau}\right)\right]^{1/2}. \tag{32}$$

The noncentrality parameter δ and the tolerance limit factor k_T depend on the parameter τ, which is unknown. Consequently it is natural to replace the parameter τ by a suitable estimator. Let $\hat{\tau} = F_\eta Z$, where Z denotes the mean square ratio s_B^2/s_w^2, and F_η the $100\eta^{th}$ percentile of an F-distribution with degrees of freedom $\nu_1 = I(J-1)$ and $\nu_2 = I - 1$. We next study the relationship between η and (I, J, R).

Substituting the estimator $\hat{\tau}$ into (30), we set

$$\begin{aligned} \hat{k}_T(\eta) &= \frac{t_{IJ-1}(\hat{\delta}, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1)\frac{s_B^2}{s_w^2} + I(J-1)F_\eta Z} \\ &= \frac{t_{IJ-1}(\hat{\delta}, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1) + I(J-1)F_\eta} \cdot \frac{s_B}{s_w} \end{aligned} \tag{33}$$

The properties of the coverage rate,

$$\gamma(\eta) = P_{\hat{\mu}, T, \hat{\tau}}\{\hat{\mu} - \hat{k}_T(\eta)s_w \le u_p\}, \tag{34}$$

was studied by simulation for various values of I, J, η. Specifically, the coverage rate was generated for 10,000 simulations for $P = .90, \gamma = .95;\quad I = 4, 8;$ $J = 3, 5, 7, 9, 13;$ $\eta = .50, .60, .70, .85$. For the sake of brevity, we provide only a summary of the conclusions.

$\gamma(\eta)$ is less than γ for $\eta = .50, .60, .70$, except for small values of R. $\gamma(\eta)$ is larger than γ for $\eta = .85$, confirming the conservative property of the Mee-Owen procedure. For fixed I and J, $\gamma(\eta)$ decreases as R increases and increases with I for fixed J and R.

A second set of simulations was carried out to ascertain the behavior of η as a function of I, J and R. For $P = .90$, $\gamma = .95$, 10,000 replications were calculated for $I = 4$ and various values of R form 0.00 to 20.00 and $J = 3, 5, 7, 9$. It was noted that η exhibits very regular behavior with a maximum (for the selected values) of .84. Thus regularity will be subsequently exploited.

We now describe the determination of η. Let I and J be fixed. If $R = 0$, $\tau = 1$ and the noncentrality parameter $\delta = \sqrt{IJ}K_p$. Hence the tolerance limit

(31) is given by

$$\hat{\mu} - t_{IJ-1}(\delta, \gamma) s_t / \sqrt{IJ}, \tag{35}$$

where s_t^2 is defined in the ANOVA table (Table 1). That is, if $R = 0$, there is no variation between batches and the model reduces to a normal random sample of size IJ. Then (35) is the tolerance limit obtained by Owen [1963].

However, we have assumed that R is unknown and will still require an estimate of τ. Since Z has the F-distribution with $I-1$ and $I(J-1)$ degrees of freedom, when $R = 0$, we determine η by

$$\eta = P\{Z \leq 1\}. \tag{36}$$

Next, we determine η for large values of R ($R \longrightarrow \infty$). Writing (25) as

$$\delta = \sqrt{IJ} K_p \sqrt{\frac{1}{J}\left(1 + \frac{J-1}{\tau}\right)}, \tag{37}$$

Since $R \longrightarrow \infty$ implies $\tau \longrightarrow \infty$, $\delta \longrightarrow \delta^* = \sqrt{I} K_p$. Then (31) becomes

$$\begin{aligned} &\hat{\mu} - \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1)\frac{s_B^2}{s_w^2} + I(J-1) F_\eta Z} \cdot s_w \\ &= \hat{\mu} - \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ(IJ-1)}} \sqrt{(I-1) + I(J-1) F_\eta} \cdot s_B \quad . \end{aligned} \tag{38}$$

We can now solve for F_η using (24) and (38), determining

$$P_{Y, s_B} \left\{ \frac{Y + \delta}{s_B / \sigma_B} \leq \frac{t_{IJ-1}(\delta^*, \gamma)}{\sqrt{IJ-1}} \sqrt{(I-1) + I(J-1) F_\eta} \right\} \geq \gamma, \tag{39}$$

where $Y = \sqrt{IJ}(\hat{\mu} - \mu)/\sigma_B$, $\delta = \sqrt{IJ} K_p \sigma_x / \sigma_B$. Y has the standard normal distribution and $(I-1)s_B^2/\sigma_B^2$ has the chi-square distribution with $I-1$ degrees of freedom. Then $(Y+\delta)/(s_B/\sigma_B)$ has the noncentral t distribution with $I-1$ degrees of freedom and noncentrality parameter δ.

Writing

$$\int_{-\infty}^{c} f_{I-1}(t; \delta) dt = \gamma \tag{40}$$

where $f_{I-1}(t;\delta)$ is the probability density function of the noncentral t distribution with $I-1$ degrees of freedom and noncentrality parameter δ. Then F_η can be determined by solving (40) for c.

Consequently we write

$$t_{I-1}(\delta,\gamma) = \frac{t_{IJ-1}(\delta^*,\gamma)}{\sqrt{IJ-1}}\sqrt{(I-1)+I(J-1)F_\eta}, \tag{41}$$

where $t_d(\delta,\gamma)$ denotes the $100\gamma^{th}$ percentile for a noncentral t distribution with d degrees of freedom and noncentrality parameter δ. As $R \longrightarrow \infty$, $\sigma_x/\sigma_B \longrightarrow 1/\sqrt{J}$, we have $\delta \longrightarrow \delta^*$. Therefore

$$F_\eta \longrightarrow F_\eta^* = \frac{IJ-1}{I(J-1)}\left(\frac{t_{I-1}(\delta^*,\gamma)}{t_{IJ-1}(\delta^*,\gamma)}\right)^2 - \frac{I-1}{I(J-1)}. \tag{42}$$

Regarding F_η as the $100\eta^{th}$ percentile of the F distribution with $I(J-1)$ and $(I-1)$ degrees of freedom, η is ready available. We now proceed to determine η for all R, $0 \leq R < \infty$. For various I, J values and $P = 0.9, \gamma = 0.95$, η values were simulated for $R = 0, .1, 1, 5, 40$. $R \longrightarrow \infty$ and $R = 0$ were determined analytically. The relationship,

$$\eta = \eta_\infty - (\eta_\infty - \eta_0)\exp(-\theta R^\zeta), \tag{43}$$

where η_∞ and η_0 are the η values when $R = \infty$ and $R = 0$ respectively. (43) satisfies the plots of the simulations with negligible error. The parameters θ and ζ can be determined by a curve fitting algorithm. Some η values for $P = .90, \gamma = .95$ are listed in Table 2. Simulation study shows that linear interpolation in η would give satisfactory results for values of I, J and R being not tabulated.

We now proceed to a verification of the efficiency of this procedure and a comparison with the Mee-Owen [1983] method.

Table 2. η values for $P = .90, \gamma = .95$

		I						
R	J	2	3	4	5	6	8	15
0	2	.58	.54	.52	.52	.51	.51	.50
	4	.64	.60	.57	.56	.55	.54	.53
	6	.66	.61	.59	.57	.57	.55	.54
	8	.67	.62	.59	.58	.57	.56	.54
	16	.67	.62	.50	.59	.58	.57	.55
	32	.68	.63	.60	.59	.58	.57	.55
.1	2	.63	.63	.62	.61	.60	.59	.55
	4	.63	.63	.63	.61	.60	.59	.56
	6	.67	.63	.63	.63	.61	.60	.57
	8	.68	.65	.63	.63	.62	.61	.61
	16	.74	.68	.67	.66	.64	.64	.63
	32	.78	.71	.70	.71	.66	.66	.63
1	2	.75	.75	.74	.72	.70	.71	.64
	4	.80	.78	.76	.74	.74	.72	.71
	6	.83	.79	.79	.77	.76	.74	.72
	8	.84	.80	.79	.77	.76	.75	.73
	16	.85	.82	.80	.78	.76	.76	.73
	32	.86	.82	.80	.79	.78	.78	.73
5	2	.84	.81	.79	.79	.76	.76	.70
	4	.87	.84	.82	.80	.78	.76	.76
	6	.89	.84	.83	.81	.80	.78	.76
	8	.89	.84	.83	.81	.81	.79	.77
	16	.89	.85	.84	.82	.81	.80	.77
	32	.89	.85	.85	.83	.82	.81	.77
∞	2	.86	.83	.81	.80	.78	.77	.74
	4	.88	.86	.83	.82	.81	.79	.77
	6	.89	.86	.84	.83	.82	.80	.78
	8	.89	.87	.85	.83	.82	.81	.79
	16	.90	.87	.85	.84	.83	.82	.79
	32	.90	.87	.86	.84	.83	.82	.79

Table 3. Simulated Coverage for $P = .90,\ \gamma = .95$
(10,000 observations)

	Mee-Owen		Present	
η	max	min	max	min
0	.990	.965	.952	.948
.1	.988	.961	.952	.948
1	.978	.948	.952	.948
5	.965	.946	.952	.948
40	.965	.948	.954	.948

3 Comparison of Tolerance Limits

Extensive numerical computations were made for the purpose of evaluating the present procedure and providing comparisons of the coverage and the actual tolerance limit with the Mee-Owen [1983] technique. Table 3 and 4 exhibit the results.

The Mee-Owen procedure as noted in the preceding section is most conservative for low values of R, particularly for small values of I and J. The coverage of the present procedure is particularly uniform across the range of values where the sampling error of the simulation is taken into account. Finally, their tolerance limits are also substantially higher for the present procedure. Some sample values are given below.

Note that the Mee-Owen procedure is less conservative for large values of R. The standard errors of the tolerance limits have also been calculated. The present procedure has been shown to have consistently smaller standard errors over most of the range of values and about the same standard error as the Mee-Owen procedure for large R.

Table 4. Tolerance limits for $P = .90, \gamma = .95$

R	I	J	Mee-Owen	Present
0	2	2	−14.11	−4.57
	2	8	−8.33	−2.33
	8	2	−2.14	−2.01
	8	8	−1.67	−1.60
.1	2	2	−15.14	−5.16
	2	8	−11.30	−3.16
	8	2	−2.27	−2.15
	8	8	−1.84	−1.76
1	2	2	−21.38	−10.62
	2	8	−21.40	−11.34
	8	2	−3.26	−3.17
	8	8	−3.00	−2.88
5	2	2	−38.05	−30.43
	2	8	−39.04	−34.47
	8	2	−5.96	−5.93
	8	8	−5.83	−5.77
40	2	2	−97.57	−99.75
	2	8	−99.85	−105.79
	8	2	−15.93	−15.95
	8	8	−16.02	−15.82

4. Numerical Illustration.

To provide a numerical comparison with the results of Lemon (1977) and Mee and Owen (1983), we will apply the present procedure to data analyzed in the above papers. Lemon's data consisted of five batches with six observations per batch. The data and specifications for the tolerance limit are summarized as follows.

$$P = .9,\ \gamma = .95,\ \hat{\mu} = 186,\ s_b = 6.87,\ s_w = 5.86,\ s_x = 9.04,$$

hence $\hat{R} = 1.37$.

Lemon obtained a lower tolerance limit of 156.3. The procedure of Mee and Owen resulted in a lower tolerance limit of 160.0, which is less conservative than Lemon's procedure. The present methodology utilizes the non-central t distribution with non-centrality parameter $\delta = 3.56$ and 29 degrees of freedom, resulting in a lower toleraance limit of 165.0, which is less conservative than the result of Mee and Owen..

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