An Almost Natural Application of Bayesian Statistics in Packaging Quality Control

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Abstract

Lower cost homogeneous items are advertised and sold by quantity in a single package across all business sectors, from food service to medical to manufacturing. Quality control measures come at a price that is somewhat exacerbated by operating in a low cost, high quantity market. It is therefore of interest to find a cost-effective way of ascertaining the number of items in a package.

Straightforward application of the central limit theorem and the Bayes' theorem allows us to find the distribution of the number of items in a package conditioned on the weight of the package, provided a prior distribution of the number of items in the package is known. Initial and updated prior distribution estimates can be formed via manufacturer's reliability data and sampling data. This information combined with a standard quality control measure of package weight provides an almost natural Bayesian framework. We will show that well-established Bayesian techniques and testing can be employed to provide a useful tool which addresses this problem.

1 Introduction

In today's highly competitive business environment, the mass packaging of homogeneous items is a common practice whether it is done by hand or by an ultra precise computer controlled machine. Both the manufacturer of the items and the customer are interested in ensuring that the proper number of items are contained in each package. From the manufacturer's point of view, she needs to implement the necessary quality control measures at the lowest possible cost. The customer may be interested in economical and efficient ways of determining if each package contains at least the contracted amount.

Most of us have seen a jar full of pennies–presumably in thousands–and wondered how many pennies were in it. One sensible solution in a situation like this would be to weigh the jar and try to estimate how many pennies are in the jar based on the weight. We propose that one way of tackling the quality assurance problem described in the previous paragraph is simply by weighing the package and incorporate the information from the distribution of the weights of the items. Even as both frequentist and Bayesian analyses may be employed in the situation, the availability of the prior information and other factors make the situation almost ideal for the Bayesian framework to be used in.

In this report, we concentrate on the situation where a customer is interested in ways of accepting or rejecting a package based on the measurement of its weight. In the next section, both frequentist and Bayesian models are constructed for the problem. In the subsequent section, some numerical results are presented.

2 Modeling

2.1 Motivation and Development of a Bayesian Model

Let N be the number of items; X_i the weight of the ith item; $W_K = \sum_{i=1}^{K} X_i$. If a single package is given and its weight is measured, then we may treat N as a parameter, and we have one observation $W_N = w_n$. A corresponding frequentist approach as well as a Bayesian approach conditioned on the observed weight can be employed in the situation.

In many practical applications, n should be at least in hundreds, if not in thousands. Whether or not X_i is normally distributed, W_n should be normally distributed by Central Limit Theorem. That is,

$$W_K | K = n \quad \sim N(n\mu, n\sigma^2) \tag{1}$$

where $\mu = E(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1)$. To facilitate the analytical treatment, we will assume from this point on that $X_1 \sim N(\mu, \sigma^2)$ and that both μ and σ^2 are known. (We also assume that the measurement is that of net weight: the tare weight has been subtracted from the raw measurement.)

Clearly, $\frac{W_K}{\mu}$ is an unbiased estimator of n. This is also the solution many will arrive at in the problem of counting the number of pennies in a jar. At this point, let us ask the following question: what are the sources of randomness in the situation? First, there is the item to item difference in weight, which leads us to model the weight of individual item as a random variable X_i. The packaging mechanism is another source of variation. So N is a proper random variable with a probability structure, and we need to model it as such. For the situation we are concerned with in this report where only the weight of one package is measured and used to estimate the number of items in it, we then have $N = n$ for a given box and wish to estimate n, and the probability distribution of N may be used as a prior distribution in a Bayesian framework. If a measurement yields the weight w_n, we can write the probability mass function of n conditioned on the measured weight w_n as follows:

$$p(n|w_n) \propto f(w_n|n)\pi(n)$$

That is,

$$p(n|w_n) = \frac{f(w_n|n)\pi(n)}{\sum_j f(w_n|j)\pi(j)} \tag{2}$$

Since n, the number of items, is discrete, the marginal of w_n is obtained by summation. Note also that the index n in w_n is not to be changed from a term to another term in the summation since w_n denotes the observed weight of the package. Treating the conditional *pmf* in Eq.(2) as the posterior in a typical Bayesian framework, we may employ any of widely used Bayesian inferential tools that would suit our purpose.

We should also make one remark here concerning any conjugate prior. The key feature of our model is in Eq.(1). The "parameter" n appears both in the mean and the variance of the normal distribution, causing n to appear both in and out of the exponential such that the well known result that the conjugate relation between normal distributions cannot be applied here even if we were to allow n to be continuous for the sake of technical convenience.

2.2 Bayesian Hypothesis Testing Using Generalized 1-0 Loss

The case we are interested in, the case where the customer is going to either accept or reject the shipment based on the weight of the box, is clearly the one where we can apply the hypothesis testing method. The customer has an obvious interest in

$$\begin{aligned} H_0 &: n \geq n_0 \\ H_1 &: n < n_0 \end{aligned}$$

where n_0 is the contracted amount. One could easily construct the hypotheses of interest for the manufacturer as well, and the following can be just as easily adapted.

In a Bayesian framework, it makes perfect sense to ask what the probability a hypothesis is correct is, since we have, in our model, probability distribution for the parameter(s) which the hypotheses are describing. Given the posterior distribution found using Eq.(2), we can compute the posterior probability that each hypothesis is correct. That is,

$$P(H_0|w_n) = \sum_{n \geq n_0} p(n|w_n) \tag{3}$$

$$P(H_1|w_n) = \sum_{n < n_0} p(n|w_n). \tag{4}$$

Choosing the hypothesis with the higher posterior probability corresponds to the Bayes' rule using 0-1 loss. In many situations, however, the customer may have different loss for different types of error. We can incorporate that information by using a generalized 0-1 loss function. The generalized 0-1 loss function allows us to incorporate differing loss based on the type of error (I or II) whereas the usual 0-1 loss assigns the equal loss on both types of error. Let C_I be the cost the customer will incur when she rejects a package with the acceptable number of items, $n \geq n_0$–that is when she makes the type I error–and let C_2 be the cost when she accepts the package when it does not contain the acceptable number of items, $n < n_0$, corresponding to the type II error. It is a well known result that rejecting the null hypothesis when the posterior probability that it is correct, $P(H_0|w_n)$, is less than $\frac{C_{II}}{C_I+C_{II}}$ (or alternatively $P(H_1|w_n) > \frac{C_I}{C_I+C_{II}}$) corresponds to a Bayes rule, having the optimality properties that go with being a Bayes rule (see Casella and Berger (1990) or Berger (1993)).

3 Numerical "Sensitivity" Analysis

Using the decision rule described in the last section, to reject the shipment when $P(H_1|w_n) > \frac{C_I}{C_I+C_{II}}$, we know it will be a "good" decision in theoretical terms knowing that it is a Bayes rule. In practice, μ and σ^2 will also vary from one setting to another, and a user of the method may be interested in how the method performs under such different settings. We have of course performed the usual posterior sensitivity analysis to see if prior misspecification affects the inference in significant manner. The numerical investigation of how the method performs under these different settings was done using Microsoft Excel. We tabulated the values of $P(H_1|w_n)$ for various settings. The loss function we used corresponded to the case where $C_I = 9C_{II}$ so that the value we are comparing the posterior probability to is $\frac{C_I}{C_I+C_{II}} = .9$. The values of $P(H_1|w_n)$ in bold red then means that the shipment is rejected.

Suppose now $n_0 = 200$, $\mu = 3$. We used a triangular prior on 190 and 240. We varied the mode and σ^2. Table 1 shows the posterior probabilities and the decisions based on them when the observed weight was 605. The robustness in prior specification is demonstrated by the proposed method giving mostly consistent decision in the same rows. In Table 2, we looked at the posterior

		Mode of the prior								
	0.9696	193	198	204	209	215	221	226	232	237
% change in σ^2	1.00 %	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	2.00 %	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4.00 %	0.9904	0.9904	0.9920	0.9920	0.9920	0.9920	0.9920	0.9920	0.9920
	6.00 %	0.9286	0.9286	0.9408	0.9408	0.9408	0.9408	0.9408	0.9408	0.9408
	8.00 %	0.8756	0.8756	0.8966	0.8968	0.8968	0.8968	0.8968	0.8968	0.8968
	10.00 %	0.8529	0.8529	0.8774	0.8781	0.8781	0.8781	0.8781	0.8781	0.8781
	12.00 %	0.8485	0.8484	0.8731	0.8744	0.8744	0.8744	0.8744	0.8744	0.8744
	14.00 %	0.8524	0.8520	0.8755	0.8777	0.8777	0.8777	0.8777	0.8777	0.8777
	16.00 %	0.8597	0.8589	0.8807	0.8838	0.8838	0.8838	0.8838	0.8838	0.8838
	18.00 %	0.8679	0.8666	0.8868	0.8907	0.8908	0.8908	0.8908	0.8908	0.8908
	20.00 %	0.8761	0.8742	0.8929	0.8976	0.8977	0.8977	0.8977	0.8977	0.8977
	22.00 %	0.8839	0.8812	0.8986	0.9040	0.9042	0.9042	0.9042	0.9042	0.9042
	24.00 %	0.8910	0.8877	0.9039	0.9099	0.9102	0.9102	0.9102	0.9102	0.9102
	26.00 %	0.8974	0.8934	0.9087	0.9152	0.9157	0.9157	0.9157	0.9157	0.9157
	28.00 %	0.9031	0.8986	0.9130	0.9199	0.9207	0.9207	0.9207	0.9207	0.9207
	30.00 %	0.9082	0.9031	0.9169	0.9242	0.9252	0.9252	0.9252	0.9252	0.9252
	32.00 %	0.9128	0.9072	0.9204	0.9280	0.9292	0.9293	0.9293	0.9293	0.9293
	34.00 %	0.9168	0.9109	0.9235	0.9315	0.9329	0.9330	0.9330	0.9330	0.9330
	36.00 %	0.9204	0.9142	0.9264	0.9346	0.9363	0.9364	0.9364	0.9364	0.9364
	38.00 %	0.9236	0.9171	0.9290	0.9373	0.9393	0.9395	0.9395	0.9395	0.9395
	40.00 %	0.9265	0.9198	0.9314	0.9398	0.9421	0.9424	0.9424	0.9424	0.9424

Table 1

robustness for different values of μ, and similar robustness is demonstrated.

Mode of the prior

% diff in w from the nμ

0.97	193	198	204	209	215	221	226	232	237
-3.00%	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-2.00%	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
-1.00%	0.9902	0.9899	0.9873	0.9873	0.9873	0.9873	0.9873	0.9873	0.9873
-0.50%	0.7975	0.7973	0.7712	0.7712	0.7712	0.7712	0.7712	0.7712	0.7712
-0.25%	0.5659	0.5659	0.5392	0.5392	0.5392	0.5392	0.5392	0.5392	0.5392
0.00%	0.4358	0.4358	0.4360	0.4360	0.4360	0.4360	0.4360	0.4360	0.4360
0.25%	0.5550	0.5550	0.5802	0.5802	0.5802	0.5802	0.5802	0.5802	0.5802
0.50%	0.7872	0.7872	0.8102	0.8102	0.8102	0.8102	0.8102	0.8102	0.8102
1.00%	0.9891	0.9891	0.9913	0.9913	0.9913	0.9913	0.9913	0.9913	0.9913
2.00%	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3.00%	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2

4 Conclusion

We have proposed a method of assuring or testing the number of items in a package only by its weight measurement and the knowledge of the distribution of the weights of the individual items. The robustness of the Bayes' rule obtained using a generalized 0-1 loss function was demonstrated.

The case we examined in this report is where the testing is done on a box by box basis. We are currently working on an extension of the method which can be phrased in the following question: Suppose we are given the weight measurements of m packages. These packages are supposedly packed with the same mechanism/procedure so that these may be modeled as *iid* random variables.

From these m measurements of weights, can we find out about the marginal distribution of n, the amount the packing mechanism puts in each box? This could be used to test the reliability of the packing mechanism quickly (without counting items one by one) and also to better specify the prior that is to be used in the case we looked at in this report.

References

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