On Optimal System Design Under Reliability and Economic Constraints[1]

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Abstract

Reliability Economics is a field that can be defined as the collection of all problems in which there is tension between the performance of systems of interest and their cost. Given such a problem, the aim is to resolve the tension through an optimization process that identifies the system that maximizes some appropriate criterion function (e.g. expected lifetime per unit cost). In this paper, we focus on coherent systems in n independent and identically distributed (iid) components and mixtures thereof, and characterize both a system's performance and cost as functions of the system's signature vector (Samaniego, IEEE-TR, 1985). For a given family of criterion functions, a variety of optimality results are obtained for systems of arbitrary order n. The case of an underlying exponential distribution is used to illustrate these results. Approximations are developed and justified when the underlying component distribution is unknown. In the latter circumstance, assuming that an auxiliary sample of size N is available on component failure times, the asymptotic theory of L-estimators is adapted for the purpose of proving the consistency and asymptotic normality (as $N \to \infty$) of estimators of the expected ordered failure times of the n components of the systems under study. These asymptotic results lead to the identification of optimal systems relative to a closely approximated criterion function. Proofs of the results stated herein appear in a referenced Technical Report.

I. Introduction

The emerging field of Reliability Economics (RE) is perhaps most easily defined by its goals rather than by its tools or results. The literature in Reliability Economics is quite widely scattered, and the area is yet to be unified and conceptualized as a distinct field of study. Roughly speaking, the field can be thought of as the collection of problems and frameworks in which there is tension between the performance of a group of systems of interest and their cost. In general, expensive systems perform quite well and inexpensive systems perform less well. Ideally, one would like to select the system that represents the best compromise between performance and cost. This is, in fact, often the goal of an RE analysis, though there are other goals of possible interest.

When one thinks of a particular manufactured system that one might consider purchasing, two questions naturally arise: (1) “How well does it work? and (2) “How much does it cost?” These questions are so natural that situations in which one or the other question might be deemed irrelevant would seem to be both extreme and quite rare. If money were truly “of no object”, then naturally one would purchase the system with the best performance, or if money was very tight, one might be forced to buy the least expensive system available without questioning its performance characteristics. Excluding these extreme situations, the natural strategy in procurement situations is to take both performance and cost into account. It is thus quite surprising that the mathematical and statistical underpinnings of doing so in a systematic way are, at present, in a relatively primitive state.

Exceptions exist in selected problem areas such as “warranty analysis” (see, for example, Blischke and Murthy (1996) and Singpurwalla (2004)), but general developments in Reliability Economics are at

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present quite sparse. In military acquisitions, for example, it is frequently the case that a particular prototype system is developed to meet certain performance and suitability goals, and that once a system meeting those goals is developed and is validated through operational testing, the system is purchased in whatever numbers the allocated budget can accommodate. Such an approach foregoes the formal investigation of optimality questions such as "What system design would give us the best performance per unit cost?" In this study, our goal is to address the question: "Is it possible to identify system designs which are optimal in some appropriate sense in the face of economic constraints?" In more common parlance, can one find the system that gives us the most bang for the buck? We discuss below our progress toward answering that question.

Let us first examine why the problem of answering the type of question posed above has heretofore resisted clean, analytical solutions. Consider the notion of "coherent systems of order (or size) n", a fundamental idea in reliability dating back to the seminal paper by Birnbaum, Esary and Saunders (1961). Coherent systems of order n are n-component systems that are monotone (i.e., the state of the system can only stay the same or improve when a component is improved), and in which every component is relevant (i.e., it actually affects system performance under some configuration of the functioning or failure of the other components).

Identifying the exact number of coherent systems of a given order is a fascinating open question. A few crude approximations exist, but all that is really known is that the number is finite but tends to be very large. The number is known to grow exponentially with n, so that, for example, there are well over a billion different coherent systems of order 30. This provides part of the explanation for the resistance seen in attempts to optimize relative to the class of all coherent systems of a given size. The problem is a discrete optimization problem in which the space to be searched is usually huge.

There is a second reason that finding analytical solutions to optimization problems would be difficult. That is that there has not been a tool available which summarizes the behavior of a system as a design parameter with respect to which one can optimize. The structure function φ (see, for example, Barlow and Proschan (1981)) which characterizes a system by the relationship between the n-dimensional vector of 1s and 0s representing the states of the n components (working or not) and the state of the system (1 or 0) is (i) awkward to compute for complex systems and (ii) too clumsy to use as an index for all coherent systems of a given size.

These two difficulties, together, have led to the reliance on "searching techniques" for seeking good (near-optimal) solutions as efficiently as possible. Genetic algorithms appear to be the favored approach in the recent literature. There is a substantial literature on the latter approach. Chapter 7 of the monograph by Kuo, Prasad, Tillman and Hwang (2001) discuss the algorithmic approach to constrained optimization problems in reliability and provide many references. For a concrete example of the use of genetic algorithms in searching for a system design that minimizes costs while achieving a fixed reliability threshold, see Deeter and Smyth (1998). An example of the use of a genetic algorithm in searching for a cost-optimal maintenance policy may be found in Usher, Kamal and Sayed (1998).

We now turn no to a discussion of some background ideas and results which provide the foundation for the approach we will take to problems of optimal system design under reliability and economic constraints.

2. Signatures and mixed systems.

In the formulation of problems in reliability economics we have studied, both of the obstacles above have been overcome, one, quite curiously, by making the space of systems of interest even larger and the other by identifying a new and useful index of that space. We'll discuss the latter issue first, as the former one follows upon it naturally. In a paper in the IEEE Transaction on Reliability, Samaniego (1985) defined the notion of "system signature". In brief, if one restricts attention to systems of order n whose components have independent and identically distributed (iid) lifetimes, then the behavior of the system's lifetime is completely determined by the underlying component lifetime distribution F and a probability vector $\mathbf{s} = (s_1, s_2,\ldots, s_n)$ called the system's "signature". The ith component of $\mathbf{s}$ is simply the probability that the ith component failure (that is, the ith order statistic among the component failure times $X_1, X_2, \ldots, X_n$) causes the system to fail. Since all n! permutations of the n failure times are equally likely under an iid assumption, the computation of **s** tends to be a manageable combinatorial problem. In Samaniego (1985), representations were given for the distribution function F_T, density function f_T, and failure rate r_T of the system lifetime T in terms of the component lifetime distribution F and the signature **s**. Although it was not applied to optimization problems at the time, it became clear over time that the signature serves as an excellent index in optimization problems involving the family of coherent systems.

The use of signatures as proxies for the corresponding system designs requires a bit of a defense. Since signatures are based on the assumption of iid component lifetimes, one might well question their relevance in problems in which components lifetimes are dependent or have different distributions. This question is taken up in Kochar, Mukerjee and Samaniego (1999), where it is argued that when comparing systems, one wants to consider a level playing field, as it is clear that a poor system with good components can outperform a good system with poor components. If, however, all components are independent and have the same distribution, then any superiority in one system's performance over another must be attributable to the system design alone. Kochar, Mukerjee and Samaniego (1999) proved a number of preservation theorem for signatures (e.g., stochastic ordering between two signatures implies stochastic ordering between the corresponding system lifetimes). These results were used to demonstrate that signatures were useful tools in the comparison of competing systems. Boland, Samaniego and Vestrup (2003) showed the notion of signature was equally applicable to comparisons among communication networks. Indeed, they established in that paper the exact relationship between Satyanarayana and Prabhakar's (1978) "dominations" and the signature of a network.

Having an index for coherent systems is, of course, not enough to render the optimization problems of interest analytically tractable. That's because one still has to contend with maximization over a large discrete space. Boland and Samaniego (2004b) proposed consideration of stochastic mixtures of coherent systems. A mixed system of order n is obtained by a randomization process over the class of coherent systems of order n. In essence, one simply picks a coherent system of order n at random according to a fixed mixing distribution. If coherent system τ_i has signature vector $\mathbf{s}^{(\mathbf{i})}$ and is chosen with probability p_i for i = 1,…, k, then it is easily seen that the mixed system $\Sigma\, p_i\tau_i$ will have signature vector $\Sigma\, p_i\mathbf{s}^{(\mathbf{i})}$. Since the k-out-of-n system (which fails upon the kth component failure) has a unit vector $\mathbf{s}_{\mathbf{k:n}}$ as its signature (with 1 as its kth element), it is clear that any probability vector **p** may be considered to be a signature. Indeed, the vector **p** is the signature of the mixed system $\Sigma\, p_k\, \mathbf{s}_{\mathbf{k:n}}$. A simple example of a mixed system or order 3 would be the system that selects the series system (whose signature is (1, 0, 0)) with probability ½ and a parallel system (whose signature is (0, 0, 1)) with probability ½. The signature of such a system is (½, 0, ½), which is different from any of the distinct signatures of the 5 possible coherent systems of order 3. Mixing clearly expands the space of available systems.

The mathematical effect of introducing mixed systems is that a complex discrete optimization problem is immediately converted into a continuous problem. We may now seek to optimize with respect to the simplex of probability vectors in an n-dimensional space. The tools of differential and integral calculus can now be brought to bear on this problem. Interestingly, we also discover, as will be explained below, that there are problems in which a certain mixed system will dominate all other systems, that is, the strategy of randomizing among a collection of coherent systems can outperform the best that can be achieved by any coherent system (a degenerate mixture) alone.

3. Optimality Criteria

Implicit in the loose description of Reliability Economics above is the existence of a criterion function that depends on both performance and cost and an optimization process for maximizing the criterion function among the class of systems under consideration. In the work we report on here, we have utilized a particular class of criterion functions with two basic properties that might be considered essential in RE problems: the criterion function should vary proportionately with measures of system performance and inversely with measures of system cost. In Samaniego (1985), it was noted that if T is the lifetime of a system in iid components and with signature **s**, then the survival function of T can written as

$$P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t), \qquad (3.1)$$

where $X_{1:n}$, $X_{2:n}$,…,$X_{n:n}$ are the ordered failure times of the n components, from smallest to largest. It follows that

$$ET = \sum_{i=1}^{n} s_i EX_{i:n}. \qquad (3.2)$$

Thus, the expected system lifetime can be written as a linear combination of the elements of the signature vector. In the same vein, one could envision the expected cost of a system as a different linear combination of the components of **s**, say,

$$EC = \sum_{i=1}^{n} c_i s_i. \qquad (3.3)$$

One instance where such a linear combination arises as an appropriate representation of cost is in the "salvage model" where one assumes a fixed cost C_F for all n-component systems, and models a component cost as A and the value of a used component (salvaged from the system after the system fails) as B. Under these assumptions, the expected cost of the system is given by

$$EC = \sum_{i=1}^{n} (C_F + n(A - B) + Bi)\, s_i, \qquad (3.4)$$

which is a linear function of the elements of **s** as above. The criterion function under which the results we've obtained are derived is somewhat more general than simply the ratio of (3.2) to (3.3) or (3.4). We have sought to optimize a criterion function with would include such a ratio as a special case. Specifically, we consider, as a measure of the relative value of performance and cost, the ratio

$$m_r(\mathbf{a}, \mathbf{c}, \mathbf{s}) = \left(\sum_{i=1}^{n} a_i s_i\right) / \left(\sum_{i=1}^{n} c_i s_i\right)^r. \qquad (3.5)$$

Several remarks on (3.5) are in order. First, we note that, while setting $a_i = EX_{i:n}$ is a natural choice for the vector **a**, it is not required by our construct, and other choices might be preferred depending on the application involved. One reasonable alternative is the vector **a** with elements $a_i = P(X_{i:n} > t)$, in which case the numerator of (3.5) would simply be the system's reliability function at the point t. Further, the salvage model is but one way of motivating a sum such as $\Sigma c_k s_k$. Another would be to obtain an expert assessment of the cost of constructing a k-out-of-n system, and then set c_k equal to that cost. The justification for that choice of the vector **c** is that the mixed system represented by the signature **s** can be represented as choosing a k-out-of-n system with probability s_k and thus incurring the cost c_k with probability s_k. The expected cost of using this mixed system would be precisely $\Sigma c_k s_k$. The exponent r in (3.5) is a tuning parameter that allows one to weigh performance and cost differently. While the case when "r = 1" is of obvious interest, a large r might be required in problems in which controlling costs is essential (putting a higher value on less expensive systems) while a small r is appropriate when performance is given more weight than cost. The choice of r will vary with the application.

4. Optimality Results

Under an iid assumption on component lifetimes and under the criterion function given in (3.5), we have obtained the following results. Proofs may be found in Dugas and Samaniego (2004).

Theorem 1: When r = 1, the criterion function (3.5) is maximized by a k-out-of n system.

The result above was obtained by variational arguments. The optimal system is the k-out-of-n system with the largest ratio of a to c, that is for k such that $a_k/c_k = \max a_i/c_i$, where the maximum is taken over the values i = 1,…, n.

Theorem 2: For $r \neq 1$, the criterion function in (3.5) is always maximized by a mixture of at most two k-out-of n systems.

Theorem 2 was obtained using the tools of multivariate calculus. For each of the n(n-1)/2 possible mixed systems in contention, the best mixture of the two systems involved can be calculated in closed form. Thus, the identification of the optimal system reduces to a simple numerical comparison. It is worth noting that, when $r \neq 1$, the optimal system might be a k-out-of-n system (i.e., a degenerate mixture), but it need not be. For example, when n = 2, r =2.5, $a_i = EX_{i:n}$, F is taken to be a uniform distribution and the salvage model for costs is assumed, the mixed system with signature (2/3, 1/3) is optimal and strictly better than either of the two coherent systems of order 2.

Theorem 3. If the sequence $\{a_i/c_i, i = 1,\ldots, n\}$ is monotone, then the optimal system is a mixture of a series system and a parallel system, with the mixture being degenerate for r sufficiently large or small. For sufficiently large r, the series system is optimal; for sufficiently small r, the parallel system is optimal.

Theorems 1 and 2 above settle the question of finding the optimal system for the criterion function in (3.5) and for the case where the vectors **a** and **c** can be completely specified. Theorem 3 sheds light on specific circumstances under which a particular type of system design is optimal. Note that the exponential distribution satisfies the hypothesis of Theorem 3 when $a_i = EX_{i:n}$ and the salvage model is assumed.

5. Statistical Issues

The problem that remains to be addressed relates to facilitating the practical application of the results described above. The problem of identifying the mixed system that maximizes the criterion function m in (1.5) has been solved for any fixed values of the vectors **a** and **c** and the constant r. Now, the cost vector **c** and the tuning parameter r involve assessments on the part of the experimenter, and it is not unreasonable to assume that these values can be determined, with the assistance of experts, in a given application of interest. The vector **a**, on the other hand, is typically a function of the unknown underlying distribution F of the iid component lifetimes. The most natural choice for **a** is the vector of expected order statistics, with the ith element being given by $a_i = EX_{i:n}$ for i = 1, 2,…., n. We will hereafter assume, for concreteness, that this is the specification of the vector **a** that has been chosen. Our inferential results about **a** can be adapted without difficulty to alternative specifications of **a** which depend on other aspects of F.

The practical implementation of the methods above for identifying an optimal system design require that the vector **a** be estimated from data. Several questions arise: how should **a** be estimated? If the vector **a*** = $(a_1^*,\ldots,a_n^*)$ represent an estimate of the vector of expected order statistics from a "hypothetical" sample of size n from F, what are the properties of the "optimal system" corresponding to **a***? In the theorems below, we provide answers to these questions. First, we note that, in the iid framework that has been assumed, one can perform independent life-testing experiments on a set of N components and estimate the expected order statistics corresponding to a sample of size n (the order of the systems under consideration) using so-called L-statistics (i.e., linear combinations of order statistics) based on a random sample of size N. Moreover, since n is the fixed size of the systems under study while N can be freely chosen by the experimenter, one may assume that N is much larger than n. The large sample behavior of our estimators (as $N \to \infty$) of $EX_{i:n}$ for i = 1, 2,…., n will be of particular interest. It is easy to estimate the order statistic $a_i = EX_{i:n}$ consistently if one is willing to make simple but restrictive assumptions on the distribution function F. For example, under the assumption that F has bounded support, one can quite easily obtain consistent estimators $\{a_i^*\}$ and show that the signature of the approximately optimal system converges to that of the optimal system as $N \to \infty$. However, the assumptions to which we've alluded fail to apply to standard lifetime distributions F of practical interest. We thus turn to a more general framework for ensuring the desired asymptotic behavior of "optimal system" corresponding to our estimator **a***.

Stigler (1969, 1974) and Shorack (1969, 1972) developed the theory for the large sample behavior of L estimators under a variety of scenarios. Under various sets of assumptions, L-estimators are shown to be $\sqrt{N}$ - consistent estimators of their target parameter. Moreover, a suitably standardized version of the statistic will be asymptotically normal. The strongest versions of such results have been obtained by van Zwet (1980) and by Helmers (1982). Applying the tools and ideas of the theory of L-statistics, we have developed a viable theory for the approximation of optimal systems in problems of practical interest. The estimator of $\mu_{i:n} = EX_{i:n}$, for i = 1, 2, … n, that we propose for study is the L-statistic given by

$$\mu^*_{i:n} = \frac{1}{N} \sum_{j=1}^{N} w_{i,j} X_{j:N}, \tag{5.1}$$

where

$$w_{i,j} = N \int_{(j-1)/N}^{j/N} [\Gamma(n+1) / \Gamma(i)\,\Gamma(n-i+1)]\, u^{i-1} (1-u)^{n-i}\, du. \tag{5.2}$$

The following asymptotic results regarding $\mu^*_{i:n}$ have been established. Proofs may be found in Dugas and Samaniego (2004).

Theorem 4. If the underlying distribution F of the iid components of all mixed systems of order n has a finite second moment, then $\mu^*_{i:n} \xrightarrow{p} \mu_{i:n}$ as the size N of the auxiliary sample grows to ∞.

Theorem 5. If the underlying distribution F of the iid components of all mixed systems of order n has a finite third moment, and if F is nondegenerate, then

$$\sqrt{N}\,(\mu^*_{i:n} - \mu_{i:n}) \xrightarrow{D} Y \sim N(0, \sigma_i^{\,2}) \qquad \text{as } N \to \infty, \tag{5.3}$$

where

$$\sigma_i^{\,2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_i(F(x))\, w_i(F(y))[F(\min(x,y)) - F(x)\,F(y)]\ dxdy, \tag{5.4}$$

and

$$w_i(u) = [\Gamma(n+1) / \Gamma(i)\,\Gamma(n-i+1)]\, u^{i-1} (1-u)^{n-i} \qquad \text{for } 0 \le u \le 1. \tag{5.5}$$

The results above allow one to estimate the criterion function m_r of (3.5) with arbitrary accuracy. The continuity of the function m_r in the vector **a** ensures the convergence $m_r(\mathbf{a^*}, \mathbf{c}, \mathbf{s}) \xrightarrow{p} m_r(\mathbf{a}, \mathbf{c}, \mathbf{s})$ as $N \to \infty$ for each fixed **c** and **s**, so that the value of the criterion function m_r for the approximately optimal signature **s*** converges as $N \to \infty$ to that of the optimal signature relative to the true but unknown vector **a**.

References

Barlow, R. E. and Proschan, F. (1981) Statistical Theory of Reliability and Life Testing, Silver Springs, MD: To Begin With Press.

Birnbaum, Z. W., Esary, J. D. and Saunders, S. C. (1961) "Multicomponent systems and structures and their reliability", Technometrics, 3, 55 – 77.

Blischke, W. and Murthy, D (1996) Product Warranty Handbook, New York: M. Dekker, Inc.

Boland, P. and Samaniego, F. (2004) "The signature of a coherent system and its applications in reliability", in Mathematical Reliability: An Expository Perspective, Soyer, R., Mazzuchi, T. and Singpurwalla, N. (Editors), New York: Klewer, 1 – 30.

Boland, P., Samaniego, F. and Vestrup, E. (2003) "linking Dominations and Signatures in network Reliability Theory", in Mathematical and Statistical Methods in Reliability, Lindqvist, B. and Doksum, K. (Editors), Singapore: World Scientific, 89 - 103

Deeter, A. and Smith, M. (1998). "Economic Design of Reliable Networks", IIE Transactions, 30, 1161 – 1174.

Dugas Michael R. and Samaniego, Francisco J. (2004) "New Results on Optimal System Design in Reliability Economics", Technical Report No. 488, Dept. of Statistics, Univ. of Calif., Davis.

Helmers, R. (1982) Edgeworth Expansions for Linear Combinations of Order Statistics, Mathematical Centre Tract 105, Amsterdam: Mathematisch Centrum.

Kochar, S., Mukerjee, H. and Samaniego, F. (1999) "On the signature of a coherent system and its application to comparisons among systems", Naval Research Logistics, 507 – 523.

Kohoutek, H. (1966) "Economics of Reliability" in Handbook of Reliability Engineering and Management, Ireson, W. G. and Coombs, C. (Editors), New York: McGraw-Hill.

Kuo, W., Prasad, V. R., Tillman, F. A and Hwang, C.L. (2001) Optimal Reliability Design, Cambridge University Press.

Samaniego, F. (1985) "On the closure of the IFR class under the formation of coherent systems", IEEE Transactions on Reliability, TR-34, 69 – 72.

Satyanarayana, A. and Prabhakar, A. (1978) "New topological formula and rapid algorithm for reliability analysis of complex networks", IEEE Transactions on Reliability, TR-27, 82 – 100.

Shorack, G. (1969) "Asymptotic Normality of linear combinations of functions of order statistics", Annals of Math Stat, 40, 2041 – 2050.

Shorack, G. (1972) "Functions of order statistics", Annals of Math Stat, 43, 412 – 427.

Singpurwalla, N. (2004) "Warranty: a surrogate for reliability", in Mathematical Reliability: An Expository Perspective, Soyer, R., Mazzuchi, T. and Singpurwalla, N. (Editors), New York: Klewer, 317 - 333.

Stigler, S. (1969) "Linear functions of order statistics", Annals of Math Stat, 40, 770 – 778.

Stigler, S. (1974) "Linear functions of order statistics with smooth weight functions" Annals of Stat, 2, 676 - 693.

Usher, J.., Kamal, A. and Sayed, H. (1998) Cost optimal preventive maintenance and replacement scheduling, IIE Transactions, 30, 1121 – 1128.

van Zwet, W. (1980) "A Strong law for linear functions of order statistics", Annals of Probability, 8, 986-990.