Techniques for Sample Size

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Software is available at
http://www.stat.uiowa.edu/~rlenth/Power/

Army Conference on Applied Statistics
October 22, 2004

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Power basics

Power function

Given a test T of a parameter θ (scalar or vector)

$$\pi(\theta, n, \alpha, \phi) = \Pr(T \text{ is "significant"} \mid \theta, n, \alpha, \phi)$$

where α is the significance level, n is the sample size, and ϕ represents other parameters (e.g., σ^2)

Power basics

Sample-size determination

$$n(\theta^*) = \min\{n : \pi(\theta^*, n, \alpha, \phi) \geq \pi_0\}$$

with θ^* set at a clinically [scientifically] important value of θ.
Typically, people choose $\pi_0 = .8$, $\alpha = .05$.

Two-sample *t* test of significance

- $\theta = \mu_T - \mu_C$, treatment vs. control
- $n = \{n_T, n_C\}$ (often, constrain $n_T = n_C = n$)
- $\phi = \{\sigma_T, \sigma_C\}$ (often, constrain $\sigma_T = \sigma_C = \sigma$)
- Test statistic for $H_0^s : \mu_T = \mu_C$ vs. $H_1^s : \mu_T \neq \mu_C$:

$$U_s = \hat{\theta}/\widehat{SE}(\hat{\theta}) \sim t'(\nu, \theta/SE(\hat{\theta}))$$

with d.f. ν (may be estimated, approximate)

- Power function:

$$\pi_s(\theta, n_T, n_C, \alpha, \sigma_T, \sigma_C) = \Pr(U_s < -t_{\alpha/2,\nu}) + \Pr(U_s > t_{\alpha/2,\nu})$$

Two-sample *t* test of equivalence

- Same sampling situation
- Let τ be a threshold for "smallness"
- Test statistic for $H_0^e : |\mu_T - \mu_C| \geq \tau$ vs. $H_1^e : |\mu_T - \mu_C| < \tau$:

$$U_e = \frac{\min\{\hat{\theta} + \tau, \tau - \hat{\theta}\}}{\widehat{SE}(\hat{\theta})} = \frac{\tau - |\hat{\theta}|}{\widehat{SE}(\hat{\theta})}$$

- Power function:

$$\pi_e(\theta, n_T, n_C, \alpha, \sigma_T, \sigma_C) = \Pr(U_e > t_{\alpha,\nu})$$

- This is equivalent to two one-sided t tests of size α; combined test is conservative.

Practical example

Strength of two materials

- Goals
  - Want ability to detect a 15% difference ($\theta^* = \log_e 1.15 = 0.14$)
  - A difference of less than 15% is negligible ($\tau = \log_e 1.15 = .14$)
  - Tests with $\alpha = .05$, power goal of $\pi_0 = 80$
- Pilot data on $Y = \log_e$ strength: $\sigma \approx .20$ independent of mean.
- Using GUI. . .
  - Sample size for each test
  - Graphs
  - Budget-based calculations

Just the FAQs

Most e-mail questions I get center on two issues

- Sample size for a “medium” effect (per J. Cohen books)
- Retrospective (observed) power

I have some opinions about these. . .

Retrospective power

Compute power based on. . .

- Observed effect size
- Observed SD(s)
- Same sample size and significance level

Rationale: If result is nonsignificant, is it because. . .

- Effect size is too small? $\leftarrow$ high retrospective power
- Sample size is too small? $\leftarrow$ low retrospective power

Retrospective power

Compute power based on. . .

- Observed effect size
- Observed SD(s)
- Same sample size and significance level

Rationale: If result is nonsignificant, is it because. . .

- Effect size is too small?
- Sample size is too small?
- Answer: The power is *always* small in this case (duh!)

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Retrospective power—another approach

Given. . .

- Observed effect size
- Observed SD(s)
- Same sample size and significance level

Then the outcome of the test is also known

- Recall that power = Pr{Reject H_0}
- GUI example

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T-shirt sizes for effects

Power of a two-sample *t* test depends on $d = |\mu_1 - \mu_2| \, / \, \sigma$

- Small: $d = .15$
- Medium: $d = .25$
- Large: $d = .40$

T-shirt sizes for effects

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Q. Who is the T-shirt supposed to fit?

- Human? $\sigma = 1$, say Medium : $|\mu_1 - \mu_2| = .25$ in

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- Human? $\sigma = 1$ $\quad$ Medium : $|\mu_1 - \mu_2| = .25$ in
- Hippo? $\sigma = 25$ $\quad$ Medium : $|\mu_1 - \mu_2| = 6.25$ in

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Q. Who is the T-shirt supposed to fit?

- Human? $\sigma = 1$ $\quad \text{Medium}: |\mu_1 - \mu_2| = .25 \text{ in}$
- Hippo? $\sigma = 25$ $\quad \text{Medium}: |\mu_1 - \mu_2| = 6.25 \text{ in}$
- Mouse? $\sigma = .04$ $\quad \text{Medium}: |\mu_1 - \mu_2| = .01 \text{ in}$

A definitive study

. . . is not based on generic criteria

- Specify effect size on the actual scale of measurement, based on well-considered scientific goals
- Need to know σ, approximately *
- If you can't do these things, reaching the bottom line is a matter of luck
- * Except possibly in cases where σ defines population norms

Planning a pilot study

One simple approach

- Control the probability of under-estimating N by a specified percentage P
- Percentage by which N is underestimated = percentage by which σ^2 is underestimated
- In normal case,

$$\begin{aligned}\Pr(S^2 \leq (1-P)\sigma^2) &= \Pr(\nu S^2/\sigma^2 \leq (1-P)\nu) \\ &= \Pr(Q \leq (1-P)\nu)\end{aligned}$$

where S^2 has ν d.f. and $Q \sim \chi^2_\nu$

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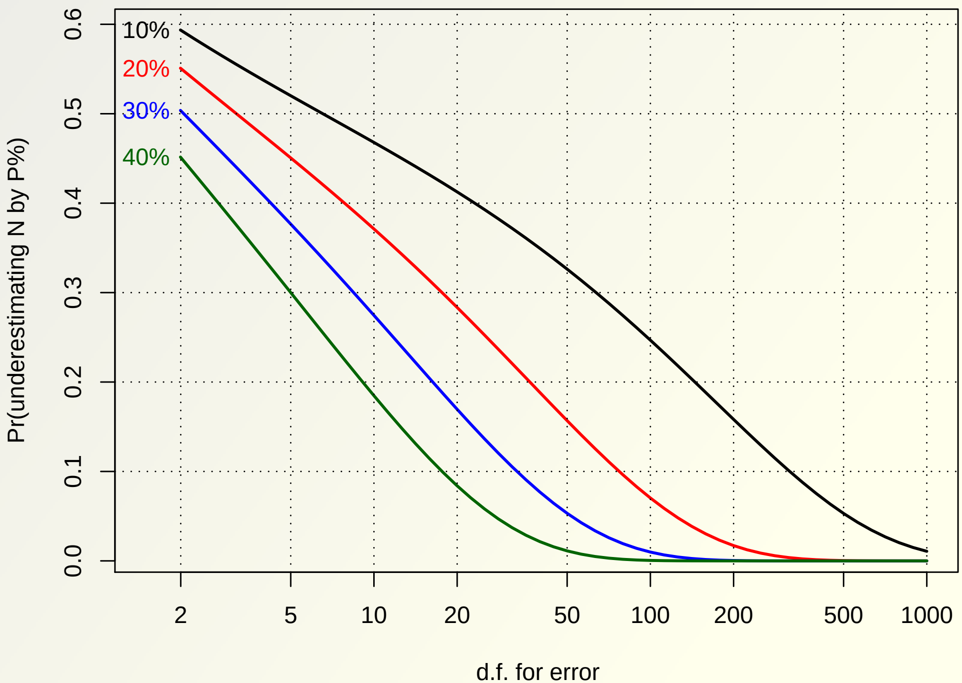
Chart for planning (or fudging)



Example: Semiconductor experiment

Structure

- Response measure: Oxide thickness of silicon wafers
- Three whole-wafer treatments
- n lots of three wafers each: one wafer per treatment
- Three sites per wafer

Target effect sizes (for power .80, .05 sig. level)

- Difference of ± 10 between two treatments
- Difference of ± 5 between two site means
- Difference of ± 15 between two treatment*site means

Available data

(From R package `nlme`) 4 lots of 3 wafers each from each of two sources; 3 sites/wafer

source × site
|
LOT
|
WAFER

```
Error: Lot
          Df Sum Sq Mean Sq F value Pr(>F)
Source     1 1830.1  1830.1  1.5261 0.2629
Residuals  6 7195.2  1199.2

Error: Lot:Wafer
          Df  Sum Sq Mean Sq F value Pr(>F)
Residuals 16 1922.67  120.17

Error: Within
            Df Sum Sq Mean Sq F value Pr(>F)
Site         2  15.44    7.72  0.6416 0.5313
Source:Site  2  58.33   29.17  2.4234 0.1004
Residuals   44 529.56   12.04
```

SD estimates

- SD(LOT) $\approx \sqrt{(1200-120)/9} \approx 11.0$ (not really needed)
- SD(WAFER in LOT) $\approx \sqrt{(120-12)/3} = 6.0 \rightarrow$ SD(LOT×treat)
- SD(ERROR) $\approx \sqrt{12} \approx 3.5$

Summary

- Power/sample size is technically messy
- Often have multiple objectives
- Sometimes need to re-define goals
- A flexible user interface can help