Seventeenth DOE Conference (1971) – Part 2 is on this CD; Part 1 is available from DTIC, Accession Database Number AD762138, $44.60.

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PROCEEDINGS OF THE SEVENTEENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH DEVELOPMENT AND TESTING

PART 2

Sponsored by
The Army Mathematics Steering Committee
on Behalf of

THE OFFICE OF THE CHIEF OF RESEARCH AND DEVELOPMENT

U. S. ARMY RESEARCH OFFICE-DURHAM

Report No. 72-2

September 1972

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Disease Severity Index*

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I. Introduction: Numerous terms of everyday life--severely ill, critical list, mild indisposition and the like--demonstrate that sickness is not a uniform condition but occurs over a broad range of severity. More recently workers studying several diseases, especially the several forms of cancer, have devised severity classifications using the term "disease staging" to facilitate both research on the disease and the choice and evaluation of treatment for particular patients.

So far these scales are wholly based on ad hoc procedures, allocation of particular patients to scale level (stages and sub-stages) are arbitrary and not always agreed on and, as yet, only limited categorization of patients and only a limited portion of the data available for a particular patient are utilized.

Beginning about twenty years ago the present writer developed a mathematical model whose purpose was to make possible estimates of a mortality dose response relation based on morbidity trials and public health statistics. In (7) this model was extended to provide an illness severity index. The actual experiment treated in that paper did not require the probability model (there called Case III) so that a subsequent search was made for data suitable for a test of that situation. Data were located on metastatic cancer patients which appeared suitable and it is

*This paper is a slight revision of reference (8) scheduled to appear. Permission to reproduce it here is gratefully acknowledged.

these data which are discussed here. However, this paper does not pretend in any sense to make a contribution to the study of cancer per se and the data are only used to reduce the theory to a convenient computational routine and to serve as a vehicle for the explanation of the rationale of the procedure. The author is of course grateful to the participating hospitals for permission to use the data and to Dr. Sidney Cutler, National Cancer Institute, for making them available. I am particularly indebted to the late Mr. George Kennedy for collaboration and inspiration in the work since its inception. Without his prodding, the current application would never have occurred. Mr. Sidney S. Spindel developed the program DISSEV explained in the appendix and performed most of the calculations arising in the study.

II. Theory of Measurement: The development of an abstract theory of mensuration seems to have begun with Hermann von Helmholz (5) in 1887. His work was in essence a formalization of the properties of a measurement procedure taken for granted in philosophical discussion and in everyday life since antiquity. Since antiquity also, two extreme attitudes towards the role of measurement and number in nature, or at least in man's understanding of nature have contended for domination. Pythagoras held that all is--or at least is comprehensible as--number; whereas Aristotle held that quality and quantity belonged to absolutely different categories (4). This alternative search for understanding, on the one hand by decomposition of current elements into more fundamental constituents and/or

properties and, on the other, by substitution for currently recognized or to be discovered phenomena of structured assemblages of yet other currently recognized components continues to this day. It appears that, so far, neither approach has exclusively won the laurels.

This purely philosphical problem did not directly lead to any profound analysis of measurement despite vigorous application of measurement everywhere throughout the Middle Ages (a form of application which included among other things forms of numerology now experiencing a vigorous revival) and despite the fact that the rise of modern science is all but universally agreed to owe much to progress in mensuration, if not indeed, to consist largely of measurement, per se. As the several social sciences developed, various special problems of measurement arose. It is curious that a psychologist (12) may have been the first to have extended an abstract approach to all forms of measurement though Campbell (2) in 1920 treated scaling, as opposed to von Helmholz's treatment of additive measurement. A comprehensive overview of the entire field has recently appeared (6).Table I is a slight adaptation of Table I of Stevens (11). The last line is taken from Suppes (13). It seems never to have previously been noticed that applied probability constitutes the construction of a fixed scale (in the sense of Table I) for categories from a nominal scale for individuals. The scale is, of course, fixed only as to its extremes, as probabilities can be expressed as per cents and vice versa.

Menger (9) has called attention to the possibility, desirability, and probable benefits of an analysis of those "formal aspects of mensuration that are shared by the application of other mathematical ideas (especially of functions and operators) to other phases of reality." A formal mathematical treatment of mensuration is given by Blakers (1).

III. Probability: The theory of probability supplies a rationale by which, given an assemblage of observations on a nominal scale, a single value of a fixed scale can be derived. For many purposes this constitutes an advantage. In the example to be discussed below, a cancer patient may or may not exhibit metastasis to a particular organ system. Knowledge of this fact aids the physician in determining how sick his patient is, which sickness level in turn determines in part his choice of treatment. It is customary to allocate cancer patients to one of four "stages" of an ordinal scale. The current technique was applied to a limited set of available data to derive a fixed (probability) scale in terms of presence or absence of metastasis to each of seven organ systems: bone, B; nodes, N; skin, S; lungs, L; pleural cavity, P; liver, V; central nervous system, C. Since any patient showing metastasis is allocated to Stage IV of the conventional scale all our distinctions are subdivisions of that stage.

Table II is a four-fold table classifying the four possible situations in which binomial probabilities apply. In a situation in which the population probabilities are known, or derived from some mathematical model, one is in a position to deduce the number of successes and failures which are to be expected in a given number of trials. This is true whether or not the probabilities are constant from trial to trial or whether they vary so long as they are known. These two situations are called "direct." In case the probabilities are known to be constant from trial to trial but the numerical value is unknown, then an actual experiment will reveal the number of successes and failures and these numbers can be used to obtain an estimate of the fixed but unknown population probability. These three situations are classic. The fourth is the subject of this paper. Before proceeding it will be necessary to introduce a further classic statistical technique.

IV. Quantal Regression: Figure (1) illustrates a statistical estimation problem which arises so frequently in a biological context that it is often called bioassay even when the application is to another field such as strength of armour plate, breaking strength of thread or any situation subject to a variable but known stress or defense against stress and outcome can be viewed as success or failure. While again multinomial outcomes are equally permissible this paper treats directly only binomial outcomes.

If a group of individuals are subject to no stress, then none will respond and the response per cent will be zero. However, as the stress level is raised, a greater and greater per cent of individuals will be affected until all respond. Per cent response is thus a monotone function of level of stress. Innumerable complications are possible, such as individuals which yield a response even in the total absence of stress. Placebo treatments are often used to adjust for this influence. Some stresses are such that some individuals are immune, so-called non-responders. All such complications are neglected in this paper though presumably to handle them would only complicate the analysis and the arithmetic in well-known ways. With-or-without the complications, any monotone relation between the variables plotted on the two axes (whether stress and per cent response or any other) can be transformed to a straight line.

To know that a relation between two variables is a straight line does not however completely characterize the relation. To do so requires that the slope and the intercept of the line be known. A full discussion of the many complications and physical interpretations arising are beyond the scope of this paper. How the slope and intercepts are determined for the application of this paper will be described in Section VI and full computing details are given in the appendix.

Since the relation between the physically defined treatment of the abscissa and the per cent response, transformed or untransformed, on the

ordinate is biunique, if a value on either axis is known, then the functional relation can be used to ascertain the corresponding value on the other axis. Thus by observing the number of animals, insects, cells, or tissue cultures which succumb to a certain treatment the strength of a chemical or drug can be measured. Because of the extreme sensitivity and high specificity of biological systems such measurements can be used at extremely high dilutions and for substances not yet characterized chemically.

If now, two such relations are plotted on the same axis, as is indeed done in Figure (1) then one of two situations necessarily occurs. Either the two curves (or several if three or more treatment-response curves are involved) have one or more points in common or they do not. The physical as opposed to the mathematical significance of intersecting dose response curves is that whereas one of the two or more responses is more frequently elicited at low stresses, (one of) the other(s) is more frequent at higher. This is simply in general implausible. Were it desirable to allow for this possibility it could in principle be done though data adequate for the simpler model (where none of the lines cross) is not easily available. When, next, a set of non-intersecting monotone relations are transformed into infinite straight lines the set of curves becomes a set of an equal number of parallel straight lines, Figure (2).

In theory we would have to know the functional form of the separate monotone functions appropriate to the abscissa and to the ordinate. In practice adequate data to unambiguously discriminate between the many closely similar candidates is never available and of little practical significance. Choice between the popular candidate metameters is a question of taste and is no different in the context of this paper than in the context of standard bioassay. I will speak of probits but others may substitute logits or whatever modification is desired.

Referring again to that cell in Table II where the probability varies from trial to trial and is in all cases unknown it is clear that if these probabilities are entirely unrelated then no series of observations however extensive can determine them since each observation introduces a new and unknown value unrelated to all past and all future observations. However, if it can be assumed that the (population) probabilities for every observation are related to each other in such a way that they fall on the system of straight lines of Figure (2) then a solution may in fact be available given sufficient data. A further basic requirement is that trial outcomes fall into sets, all members of each set being related in that all share a common value of the independent variable. In Figure (2) this is illustrated by the vertical line at s. A given patient will fall at a single point on the sickness scale, though

he may exhibit any combination of presences and absences of the n symptoms. The slope of the set of lines, each representing a particular site of metastasis, is chosen as 0.1 since it is at our disposal and doing so yields convenient two-digit values of the illness index (sickness score).

V. Individual Sickness Score: In Section IV it was explained that a plausible dose response model applicable to severity of illness and possibly also in other contexts including certain problems in pollution research consists of a relation between degree of illness measured by a sickness score, s, and the probability of occurrence of one or more all-or-none symptoms, signs, or laboratory findings. From now on I will speak entirely in terms of the one application so far made which was to the finding of metastasis in certain of seven organ systems in each of 908 breast cancer patients. I am grateful to the participating hospitals for making their records available for this study. This application is made purely to illustrate a statistical technique and its usefulness in cancer research will require much further study.

To determine how ill a given cancer patient is by this technique it is necessary to have available the equation of each of the lines of Figure (2). How, on the basis of records of cases of the disease these

equations can be derived will be the subject of the next Section. In this Section knowledge of these equations will be assumed.

It must be admitted at once that the sickness scores are not determined with high precision since the number of symptoms (in our case the number of organ systems examined for metastasis) is essentially the sample size. In turn, however, since the technique effectively summarizes all the metastasis observations into a single figure, there is no handicap in much more thorough examinations for and much more detail in recording observations--and every incentive to do so long as precision of diagnosis is sharpened thereby.

A detailed discussion of the considerations in the choice and definition of an illness index or sickness score is presented in (1). In essence it amounts to the fact that the relation between each symptom separately and the index is a bioassay equation

$$Y_i = a_i + bs \qquad (1)$$

where the intercepts a_i vary with the symptom under consideration and the slope, b, is common to all.

The sickness score, s, characterizes a specific level of disease severity. For a fixed s, the set of equations (1) determines the probability of a patient at that level of disease manifesting each of the n relevant symptoms. Since we are devising a new scale, the slope b can

be arbitrarily given any convenient value. The value 0.1 results in a score of zero for perfect health (Y = 0) and 100 for a maximal score when Y = 10 for that one symptom whose intercept a_i is zero. Again the sickness scores will be centered around 50 if the value $a_i = o$ is arbitrarily assigned to a symptom of average frequency, in our example metastasis to the lungs.

If the characteristic function x_i is chosen as unity if a particular patient exhibits the symptom i (metastasis to site i in our illustration) and zero if that organ system is free of metastasis, then the likelihood function for that particular combination of symptoms becomes

$$L = \Pi P_i^{x_i} (1-P_i)^{1-x_i} \qquad (2)$$

Each P_i is a function of Y_i which, in turn is a function of s by means of equations (1). There is therefore just one maximum likelihood equation:

$$\frac{d \log L}{ds} = o = \sum \left\{ \left(\frac{X_i}{P_i} - \frac{1 - X_i}{1 - P_i} \right) \frac{d P_i}{d Y_i} \right\} \qquad (3)$$

Because of the quantities dP_i/dY_i equation (3) is best solved by trial and error. A detailed example is given in the appendix.

In practice however this solution would be done in advance by computer for all possible symptom combinations and supplied in a table.

In our example of seven metastatic sites only 128 different values of s are possible. Use of the method in medical practice then would consist of a single table look up.

Gain to the physician would occur in two ways. First the balancing off of metastasis to certain sites and its absence in others would be objectively performed, relieving the attending physician of attempting to do so intuitively, particularly for the rarer combinations, and it would be based on all available data and not only on his own personal practice. Second, occurrence of rare combinations would be shorn of any enigmatic character and reduced to the value of a familiar parameter--the sickness index, s. This latter benefit to the physician could well outweigh the former in his attempt to rationally evaluate the condition of his patient.

VI. Construction of the Sickness Score: Section V describes how a particular set of symptoms may be used to assign a sickness score to a particular patient. Before this can be done it is necessary to know the value of the a_i in equations (1). This can only be done on the basis of accumulated experience. In the current example, records of 908 patients were available. Because the model being fitted to the data assumes that metastases to the several organ systems are ordered in a fixed order irrespective of the illness of the patient (a deduction from the parallelism of equations (1)) that order is given by the combined frequency over all records. It is: bone, 468; nodes, 408; skin, 230; lungs, 216; pleura, 179; liver, 97; and central nervous system, 35. Furthermore the per cent of 908

the true population per cent occurrence for a patient of sickness index = 50. Using these values a first estimate of each a_i in equations (1) is available. From these the records of each of the 908 patients can be used to calculate a preliminary estimate of illness for each patient. Next these 908 sickness scores can be used together with the observed per cent responses for each metastatic site--independently but with a common slope of 0.1--to calculate improved values of the intercepts a_i. All steps are done by computer and converge rapidly to stable values. Full details are given in the appendix.

VII. Check of the Model: An intuitive measure of level of illness is afforded by the duration of life remaining. In the case of metastatic breast cancer patients this period is conveniently measured in months, though in individual cases, patients may live for many years. Of course, this index is useless as a guide to treatment, but if a forecast of months of life remaining is made on the basis of a sickness score derived from extent of metastasis this forecast could indeed be useful. Post examination survival was available for 829 of the 908 patients used in deriving the index. While of course it would be desirable to apply the derived index to new metastatic cancer patients, a check based on survival of the same patients used to derive the index should be of some value since no use whatever was made of the survival information in forming the index.

There were 67 classes of patients within which all patients (ranging from one to 169) had metastasis to one or more sites but not to all, and where the sites were identical for all patients within the one class. These

patients differed widely in a number of other respects, such as grade of tumor, age, post menstrual age, and so on. Again the duration of post examination survival also differed widely within some classes but as the index depends on the pattern of metastasis, and the pattern of metastasis only, the logarithm of the geometric mean of the survival times in months for each class was fitted by weighted least squares with the sickness score, s, as abscissa. The observed and fitted values are given in detail in Table III and the results are plotted in Figure (3).

At first sight there appears to be a wide scatter about the line. However, if one notices the small numbers attached to each of the observed points which are the numbers of patients with those particular patterns of metastasis he will see that no point based on an appreciable number of patients departs widely from the line. The points of Figure (3) are given by the sickness score and logarithm of mean survival time. The larger classes mask a considerable range of variation between patients all showing metastasis to exactly the same organ systems. It is a postulate of the model that these variations are to be accounted for by <u>other</u> factors and not by the pattern of metastasis. Further that the wide departure from the line shown by many patterns of one, two, or three patients only are to be explained by the fact that these other factors had no chance to average out in such small samples.

For critical examination reference should be made to Table III. The word "pattern" designates the combination of organ systems in which

metastases were found. The specific locations are indicated in the second column. It is most important to realize that the "sickness scores" of column three depend exclusively on these patterns and on the patient counts in column four, and not at all on the observed log survival times of column five. The last column (seven) is the squared difference between the observed log survival time in column five and the value determined by the model in column six multiplied by the patient count in column four.

The internal mean square, 0.182, based on 762 degrees of freedom is a measure of the scatter between survival times of all patients showing the same pattern of metastasis. A Bartlett's Test of the 67 patterns which actually occurred showed that these variations are very homogeneous.

The residual mean square, 0.293, based on 65 degrees of freedom is a measure of how well the set of 67 logarithms of geometric mean survival times is accounted for by a linear regression on sickness score. Since the F of 1.61 is highly significant it appears that at least some variation may be due to other factors, though a parabolic relation appears unlikely. The disproportionate contributions to the residual mean squares arise from patterns number 2, 6, 14, 49, and 63. It is easy to dismiss the latter two as outliers since they are based on two and three patients respectively. That metastasis to the lymph nodes is less life threatening and to liver more so than most sites is perhaps in conformity with clinical experience. The unexpected longevity of patients with metastasis to both skin and nodes is at least favorable. My reason for this last discussion is only to

illustrate how a disease severity index, like any mathematical model, can organize a body of data and suggest problems requiring further study by its failures quite as much as solve problems by its successes.

TABLE I*

Properties of Types of Scales

Type of Scale	Empirical Operation	Permissible Transformation+	Central Tendency
Nominal	permiting	equivalence	Mode
Ordinal	ordering	monotonic	Median
Interval	differencing	affine	Arith. Mean
Ratio	dividing	similarity	Geom. Mean Harmonic Mean
Fixed**	none	identity	Per Cent

+adapted from Blakers (1)

*adapted from Stevens (4)

**taken from Suppes (5).

TABLE II

Types of Situations in which Binomial Estimation Applies

Nature of Probability	Estimation Situation	
	Direct	Indirect
Constant	Np	r/n
Variable	ΣP_i	?

TABLE III*

Regression of Log Survival on Sickness Score

PATTERN NUMBER	SYM. PAT.** BNSLPVC	SICKNESS SCORE	PATIENT COUNT	LOG SURVIVAL TIME OBS.	CALC.	SQUARED RESIDUAL
1	1000000	38	169	1.19	1.18	0.0169
2	0100000	39	85	1.27	1.16	1.0285
3	0001000	39	43	1.18	1.16	0.0172
4	0010000	40	21	1.20	1.14	0.0756
5	0000100	40	30	1.20	1.14	0.1080
6	0000010	41	12	0.71	1.12	2.0172
7	0000001	43	8	0.85	1.08	0.4232
8	1000100	45	20	1.11	1.03	0.1280
9	1100000	45	59	1.04	1.03	0.0059
10	0110000	45	51	1.07	1.03	0.0816
11	1001000	45	19	1.09	1.03	0.0684
12	0001100	45	22	1.05	1.03	0.0088
13	0101000	45	23	0.90	1.03	0.3887
14	1010000	45	27	1.24	1.03	1.1907
15	0010100	45	9	0.82	1.03	0.3969
16	0100100	45	13	0.84	1.03	0.4693
17	0011000	45	10	1.21	1.03	0.3240
18	0000110	46	1	0.95	1.01	0.0036
19	0100010	46	17	0.77	1.01	0.9792
20	0001010	46	3	0.69	1.01	0.3072
21	1000010	46	16	0.95	1.01	0.0576
22	1000001	47	6	1.02	0.99	0.0054
23	0001001	47	2	1.09	0.99	0.0200
24	0100001	47	3	0.46	0.99	0.8427
25	1011000	49	4	0.98	0.95	0.0036
26	0101100	49	5	0.70	0.95	0.3125
27	0110100	49	13	0.74	0.95	0.5733
28	0111000	49	11	1.15	0.95	0.4400
29	1001100	49	9	0.74	0.95	0.3969
30	0011100	49	2	0.89	0.95	0.0072
31	1010100	49	3	0.82	0.95	0.0507
32	1010010	50	2	1.22	0.93	0.1682
33	0011010	50	1	0.85	0.93	0.0064
34	0001110	50	2	0.73	0.93	0.0800

* For explanation see text.

** B = bone, N = lymph nodes, S = skin, L = lungs, P = pleura, V = liver, C = central nervous system.

TABLE III (Continued)

PATTERN NUMBER	SYM. PAT.** BNSLPVC	SICKNESS SCORE	PATIENT COUNT	LOG SURVIVAL TIME OBS.	LOG SURVIVAL TIME CALC.	SQUARED RESIDUAL
35	1100100	50	11	0.92	0.93	0.0011
36	1101000	50	11	0.81	0.93	0.1584
37	0101010	50	2	0.80	0.93	0.0338
38	0110010	50	5	0.65	0.93	0.3920
39	1100010	50	4	0.64	0.93	0.3364
40	1000110	50	2	0.50	0.93	0.3698
41	1110000	50	25	0.92	0.93	0.0025
42	0100110	50	1	0.30	0.93	0.3969
43	1100001	51	2	0.94	0.91	0.0018
44	0101001	51	2	0.63	0.91	0.1568
45	1000011	51	1	0.78	0.91	0.0169
46	1001001	51	1	0.30	0.91	0.3721
47	0111100	53	5	0.96	0.86	0.0500
48	1101100	54	2	1.24	0.84	0.3200
49	1011100	54	2	1.66	0.84	1.3448
50	1110010	54	3	0.74	0.84	0.0300
51	1101010	54	1	0.85	0.84	0.0001
52	1001101	54	1	1.04	0.84	0.0400
53	0111010	54	1	0.70	0.84	0.0196
54	1110100	54	2	1.00	0.84	0.0512
55	1111000	54	9	0.90	0.84	0.0324
56	1100110	54	1	0.00	0.84	0.7056
57	1100011	55	1	1.20	0.82	0.1444
58	1110001	55	1	0.60	0.82	0.0484
59	1111010	59	1	0.70	0.74	0.0016
60	1101101	59	1	1.04	0.74	0.0900
61	1110011	59	1	1.11	0.74	0.1369
62	1110110	59	2	0.57	0.74	0.0578
63	1111100	59	3	1.48	0.74	1.6428
64	1101110	59	1	0.00	0.74	0.5476
65	1110111	64	1	0.85	0.63	0.0484
66	1111101	65	1	0.00	0.61	0.3721
67	1111110	66	1	1.26	0.59	0.4489

Res. M.S. = 0.293 64 d.f. F = 1.61 r = -0.63
Int. M.S. = 0.182 762 d.f. P < 0.02

** B = bone, N = lymph nodes, S = skin, L = lungs,
P = pleura, V = liver, C = central nervous system.

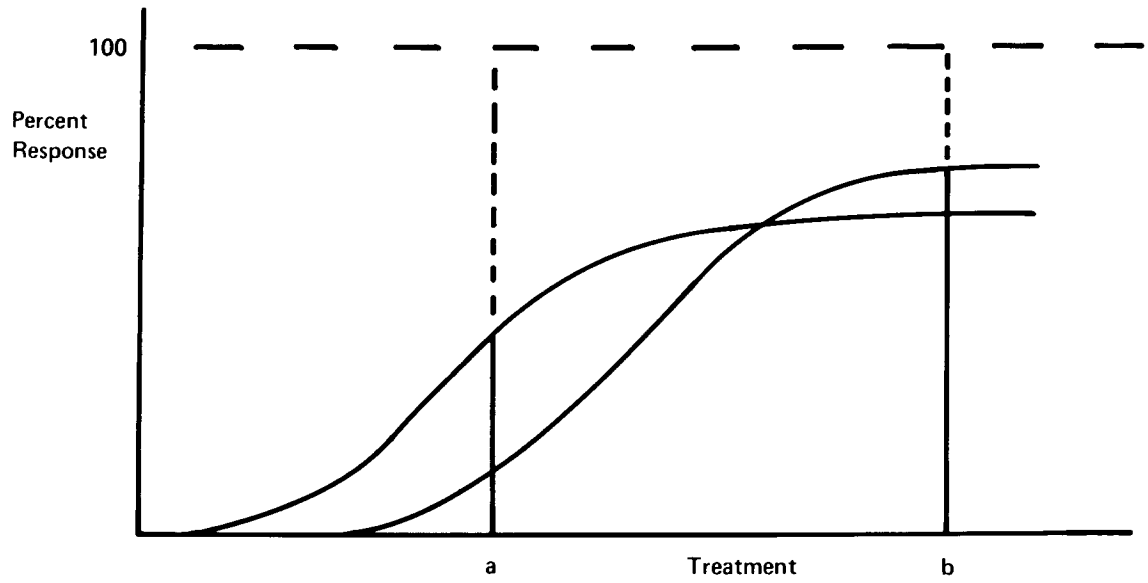

Figure 1 Hypothetical Quantal Response to Two Stresses or Treatments.

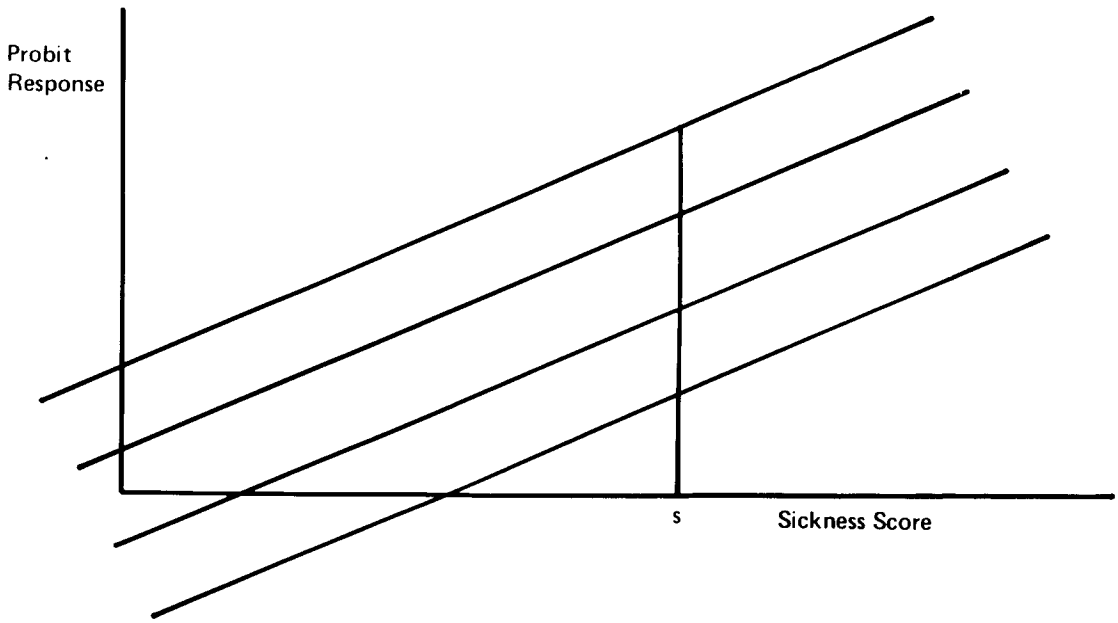

Figure2 Linear Relations between Average Frequency Response and Severity of Stress or Treatment of Several Binomial Criteria after Transformation to Suitable Scale on Each Axis.

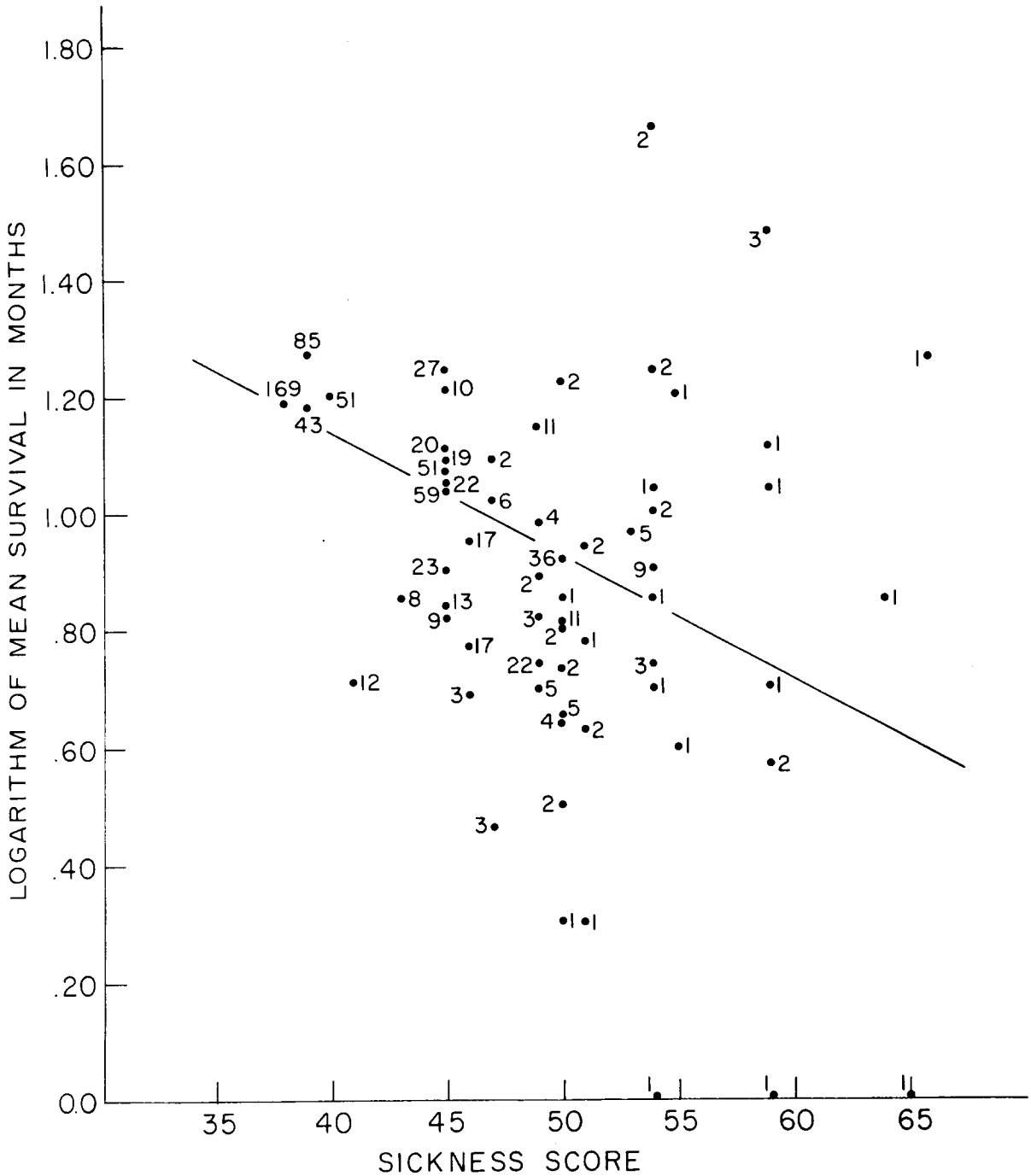

Figure 3 Sickness Score vs. Mean Survival Time
Numbers at points are numbers of patients.

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Appendix

This appendix gives full computing details for constructing and using the multiple attribute index described in the main text. Familiarity with that discussion will be assumed below. To calculate the index for a given patient requires that estimates of the parameters a_i and b of equations (1) be known. This can only be done if a fairly large body of data be available from hospital records or otherwise. In estimating the values of the a_i and b from this large body of data the calculations needed to determine the sickness index, s, must be performed repeatedly for every patient involved. Accordingly, this calculation will not be separately described but included in the full discussion of estimating the a_i and b.

Table I of this Appendix lists the seven metastatic sites in order of frequency of occurrence in 908 breast cancer patients, irrespective of presence or absence of metastasis in any other site. It will be seen that metastasis is most often found in bone and least often in the central nervous system. The observed per cent incidence of metastasis by site in these 908 patients is given in column five. It is interesting to note that the most sensitive site, bone, is attacked only slightly more often than missed in equally severely ill breast cancer patients. In column six, per cent responses are converted to probits. Column seven is obtained from column six by subtracting the entry for lung from each entry of the column. The rationale for this step is as follows: In devising a scale for a previously unquantified concept we are free to choose arbitrarily and independently two features of the scale; the interval between successive

scale values and the origin of the scale. By setting the adjusted probit for lung equal to zero we are choosing a scale origin such that one half of patients with the mid-scale level of illness severity will show metastasis to the lung. Of course, a far greater per cent will have bone, lymph node, and skin metastasis, and fewer will have metastasis to the pleural cavity, the liver, or the central nervous system. Again, the coefficient of the sickness is arbitrarily chosen equal to 0.1 so that as the sickness index, s, ranges from 0 to 100, the probit of the percentage of breast cancer patients, all equally ill, which show metastasis to the lungs will range from 0 to 10, and those whose lungs are equally likely to and not to show metastasis is 50%.

If the probit equations in the last column of Table I were population estimates, then the sickness severity index for any given pattern of presences and absences of metastasis in the seven sites studied could be ascertained by finding, by trial and error, that value of s that maximizes the likelihood of the given symptom pattern. Since with seven symptoms there are at most 128 patterns these values would be calculated once and for all and tabled for clinical use. Of course, the physician would perform no calculation--and at most a single look up. I believe however his realization that rare and "bizarre" patterns of breast cancer metastasis are <u>not</u> enigmatic, but on the contrary to be expected with a prescribable frequency could well be a more useful service than the provision of a numerical illness index.

Since the equations in the last line of Table I are not absolute but on the contrary highly provisional, of course we can do no better than to accept the data at face value and estimate parameter values by successive approximations. The procedure is illustrated in Table II, which is limited to the final cycle of determining the value of s from the provisional probit lines in Table I for the particular pattern in which metastases occur in the skin, pleura and liver but are absent from bone, lymph nodes, lungs, and the central nervous system (Table II, Column 4). The calculations of Table II are those involved in calculating a patient's sickness score on the basis of his symptoms. Once the final symptom regressions are known (Table I, Column 7) these calculations are done only once. In practice, a table would be prepared so that all calculation would be by-passed. The total for the log R column for a guessed s of 48 is less than either a slightly higher value (s = 50) or a slightly lower value (s = 45). Hence it is taken as the sickness index for this particular combination of metastatic sites for this first cycle of approximation. Accepting the probit lines of column eight of Table I we calculate the maximum likelihood value of s for every symptom pattern. All patterns which actually occur are all that is necessary but it is simpler by computer to get all of them. All patients with the same pattern are equally ill, so far as this one index is concerned. However, it is by no means necessary that the only probit lines be metastasis to given organ systems. Any or all all-or-none signs or symptoms or laboratory findings would do as well.

Presumably quantitative measures would be included by multiple regression techniques.

Having now a first estimated value for the illness index of each pattern and the number of patients in our sample with that pattern, straight-forward probit calculations yield new values for the intercepts in column seven of Table I where the fitting is done with all slopes restricted to being 0.1 and the several lines are fitted on the basis of the fraction of patients with a given score showing that symptom and disregarding metastasis to any other site. On the basis of the new probit lines replacing those in column eight of Table I new illness indices are calculated for each symptom pattern and so on until a preset degree of convergence is reached.

The successive computational steps are outlined conveniently for ready reference in Table III. The DISSEV program in the General Electric Mark II version of BASIC is given in Table IV and an explanation of the computer calculations in Table V.

TABLE I*

FREQUENCY OF METASTASIS BY SITE

908 BREAST CANCER PATIENTS**

Line	System	Sym	Nbr	%	Probit	Adj Prob[1]	Initial Equation[2]
1	Bone	B	468	51.5	5.04	0.75	Y = 0.75 + 0.1s
2	Lymph Nodes	N	408	44.9	4.87	0.58	Y = 0.58 + 0.1s
3	Skin	S	230	25.3	4.33	0.04	Y = 0.04 + 0.1s
4	Lung	L	216	23.8	4.29	0.00	Y = 0.1s
5	Pleura	P	179	19.7	4.15	-0.14	Y =-0.14 + 0.1s
6	Liver	V	97	10.7	3.76	-0.53	Y =-0.53 + 0.1s
7	Cen Nerv Sys	C	35	3.8	3.23	-1.06	Y =-1.06 + 0.1s

*See text for explanation

**I am indebted to the cooperating hospitals for these data.

[1]The probit for lung incidence of metastasis is subtracted from each value in the preceding column.

[2]The coefficient 0.1 is used so that the sickness severity score, s, can be expressed conveniently in two digits.

TABLE II

WORKSHEET FOR CALCULATION OF SICKNESS INDEX (S)[a]

LINE	Organ System	Symbol	Pattern[b]	Probit Intercept	S 45 Y	S 45 1000P	S 45 1000Q	S 45 Log R	S 50 Y	S 50 1000P	S 50 1000Q	S 50 Log R	S 48[c] Y	S 48[c] 1000P	S 48[c] 1000Q	S 48[c] Log R
1	Bone	B	N	0.75	5.25	600	400	2.6021	5.75	770	230	2.3617	5.55	710	290	2.4624
2	Nodes	N	N	0.58	5.08	530	470	2.6721	5.58	720	280	2.4472	5.38	697	353	2.5478
3	Skin	S	P	0.05	4.55	327		2.5145	5.05	520		2.7160	4.85	440		2.6435
4	Lung	L	N	0.00	4.50	310	690	2.8388	5.00	500	500	2.6990	4.80	420	580	2.7634
5	Pleura	P	P	-0.14	4.36	260		2.4150	4.86	445		2.6484	4.66	367		2.5647
6	Liver	V	P	-0.53	3.97	152		2.1818	4.47	300		2.4771	4.27	233		2.3674
7	C.N.S.	C	N	-1.05	3.45	060	940	2.9731	3.95	148	852	2.9304	3.95	146	854	2.9315
								18.1974				18.2798				18.2807

[a]For explanation see Table III and text of appendix.

[b]Pattern in which metastases occur in skin, pleura, and liver but not elsewhere.

[c]Preliminary estimate of sickness score for chosen pattern.

TABLE III

EXPLANATION OF CALCULATIONS*

IN COLUMNS OF TABLE II

Line	Col.	Heading	Explanation
1	2	Organ System	Sites examined for metastasis.
2	3	Symbol	Arbitrary symbol for site.
3	4	Pattern	Indicates whether the patient record records that metastasis did or did not occur in that organ system.
4	5	Probit Intercept	Copied from Column 7 of Table I.
5	6	Y	Calculated from the equations of Column 8, Table I by arbitrarily setting S = 45 as shown at the head of this set of four columns.
6	7	1000P	Obtained from Probit Table.
7	8	1000Q	Subtracting 1000P from 1000, since symbol in Column 3 is N.
8	9	Log R	Log of 1000Q if shown, otherwise log of 1000P.
9	10-17		Equivalent to 6-9.
10	(S is estimated as 48, since its Log R total exceeds that of S = 45 and S = 50).		

* These manual calculations are only to explain the concept. Actual calculation is performed by computer program DISSEV.

TABLE IV*
PROGRAM DISSLV

```
100 DATA 7,101,-51
110 DATA .723,.5275,-.1054
120 DATA -.1263,-.3919,-.7541
130 DATA -1.5239
140 FILES DISEV3
150 SCRATCH #1
160 DIM N(1,200),Q(1,200)
170 DIM V(200,20),B(20,1)
180 READ L,N4,G
190 MAT N=ZER(1,N4)
200 MAT Q=ZER(1,N4)
210 MAT V=ZER(N4,L)
220 MAT B=ZER(L,1)
230 LET F=.39894228
240 LET X(1)=-.774596689
250 LET X(2)=0
260 LET X(3)=-X(1)
270 LET M(1)=.555555556
280 LET M(2)=.888888889
290 LET M(3)=M(1)
300 FOR J=1 TO L
310 READ B(J,1)
320 NEXT J
330 FOR K=1 TO N4
340 FOR J=1 TO L
350 LET N(1,K)=(G+K)/10
360 LET W(J)=B(J,1)*N(1,K)
370 IF W(J)<0 THEN 400
380 IF W(J)=0 THEN 430
390 GO TO 450
400 LET W(J)=-W(J)
410 LET C1=1
420 GO TO 450
430 LET C=0.5
440 GO TO 630
450 IF W(J)>5 THEN 480
460 IF W(J)=5 THEN 530
470 IF W(J)<5 THEN 510
480 LET D=-5
490 LET E=-W(J)
500 GO TO 550
510 LET D=E=0
520 GO TO 580
530 LET C=0.5
540 GO TO 560
550 GOSUB 1000
560 LET C=C+.49999971335
570 GO TO 590
580 GOSUB 1000
590 IF C1=1 THEN 610
600  GO TO  630
610  LET C = 1 - 0
620 LET C1=0
630 LET V(K,J)=C
640 NEXT J
650 NEXT K
660 FOR K=1 TO N4
670 FOR J=1 TO L
680 WRITE #1,V(K,J)
690 NEXT J
700 NEXT K
710 PRINT
720 PRINT
730 FOR J=1 TO L
740 READ S(J)
750 PRINT S(J);
760 NEXT J
770 FOR K=1 TO N4
780 LET Q(1,K)=1
790 FOR J=1 TO L
800 IF S(J)=1 THEN 830
810 IF S(J)=0 THEN 840
820 LET B(J)=V(K,J)
830 GO TO 850
840 LET B(J)=1-V(K,J)
850 LET Q(1,K)=Q(1,K)*B(J)
860 NEXT J
870 NEXT K
880 PRINT
890 LET Q1=0
900 FOR K=1 TO N4
910 IF Q(1,K)<=Q1 GO TO 940
920 LET Q1=Q(1,K)
930 LET K1=K
940 NEXT K
950 PRINT"MAX. ML=";Q1;
960 PRINT TAB(30);"SCORE=";K1-1
970 PRINT "########"
980 PRINT
990 GO TO 730
1000 LET K2=(W(J)+D)/50
1010 LET C=A1=0
1020 LET H=K2/2
1030 FOR Z=1 TO 50
1040 LET T=Z*K2+E-H
1050 FOR J=1 TO 3
1060 LET X=X(J)*H+T
1070 LET Y=EXP(-(X↑2)/2)
1080 LET A1=A1+M(J)*Y
1090 NEXT J
1100 NEXT Z
1110  C = F*A1*H+0.5
1120 RETURN
```

* Program from General Electric BASIC Mark II.

TABLE IV*
PROGRAM DISSEV

```
1130 DATA 1,0,0,0,0,0,0
1140 DATA 0,1,0,0,0,0,0
1150 DATA 1,1,0,0,0,0,0
1160 DATA 0,0,1,0,0,0,0
1170 DATA 1,0,1,0,0,0,0
1180 DATA 0,1,1,0,0,0,0
1190 DATA 1,1,1,0,0,0,0
1200 DATA 0,0,0,1,0,0,0
1210 DATA 1,0,0,1,0,0,0
1220 DATA 0,1,0,1,0,0,0
1230 DATA 1,1,0,1,0,0,0
1240 DATA 0,0,1,1,0,0,0
1250 DATA 1,0,1,1,0,0,0
1260 DATA 0,1,1,1,0,0,0
1270 DATA 1,1,1,1,0,0,0
1280 DATA 0,0,0,0,1,0,0
1290 DATA 1,0,0,0,1,0,0
1300 DATA 0,1,0,0,1,0,0
1310 DATA 1,1,0,0,1,0,0
1320 DATA 0,0,1,0,1,0,0
1330 DATA 1,0,1,0,1,0,0
1340 DATA 0,1,1,0,1,0,0
1350 DATA 1,1,1,0,1,0,0
1360 DATA 0,0,0,1,1,0,0
1370 DATA 1,0,0,1,1,0,0
1380 DATA 0,1,0,1,1,0,0
1390 DATA 1,1,0,1,1,0,0
1400 DATA 0,0,1,1,1,0,0
1410 DATA 1,0,1,1,1,0,0
1420 DATA 0,1,1,1,1,0,0
1430 DATA 1,1,1,1,1,0,0
1440 DATA 0,0,0,0,0,1,0
1450 DATA 1,0,0,0,0,1,0
1460 DATA 0,1,0,0,0,1,0
1470 DATA 1,1,0,0,0,1,0
1480 DATA 0,0,1,0,0,1,0
1490 DATA 1,0,1,0,0,1,0
1500 DATA 0,1,1,0,0,1,0
1510 DATA 1,1,1,0,0,1,0
1520 DATA 0,0,0,1,0,1,0
1530 DATA 1,0,0,1,0,1,0
1540 DATA 0,1,0,1,0,1,0
1550 DATA 1,1,0,1,0,1,0
1560 DATA 0,0,1,1,0,1,0
1570 DATA 1,0,1,1,0,1,0
1580 DATA 0,1,1,1,0,1,0
1590 DATA 1,1,1,1,0,1,0
1600 DATA 0,0,0,0,1,1,0
1610 DATA 1,0,0,0,1,1,0
1620 DATA 0,1,0,0,1,1,0
1630 DATA 1,1,0,0,1,1,0
1640 DATA 0,0,1,0,1,1,0
1650 DATA 1,0,1,0,1,1,0
1660 DATA 0,1,1,0,1,1,0
1670 DATA 1,1,1,0,1,1,0
1680 DATA 0,0,0,1,1,1,0
1690 DATA 1,0,0,1,1,1,0
1700 DATA 0,1,0,1,1,1,0
1710 DATA 1,1,0,1,1,1,0
1720 DATA 0,0,1,1,1,1,0
1730 DATA 1,0,1,1,1,1,0
1740 DATA 0,1,1,1,1,1,0
1750 DATA 1,1,1,1,1,1,0
1760 DATA 0,0,0,0,0,0,1
1770 DATA 1,0,0,0,0,0,1
1780 DATA 0,1,0,0,0,0,1
1790 DATA 1,1,0,0,0,0,1
1800 DATA 0,0,1,0,0,0,1
1810 DATA 1,0,1,0,0,0,1
1820 DATA 0,1,1,0,0,0,1
1830 DATA 1,1,1,0,0,0,1
1840 DATA 0,0,0,1,0,0,1
1850 DATA 1,0,0,1,0,0,1
1860 DATA 0,1,0,1,0,0,1
1870 DATA 1,1,0,1,0,0,1
1880 DATA 0,0,1,1,0,0,1
1890 DATA 1,0,1,1,0,0,1
1900 DATA 0,1,1,1,0,0,1
1910 DATA 1,1,1,1,0,0,1
1920 DATA 0,0,0,0,1,0,1
1930 DATA 1,0,0,0,1,0,1
1940 DATA 0,1,0,0,1,0,1
1950 DATA 1,1,0,0,1,0,1
1960 DATA 0,0,1,0,1,0,1
1970 DATA 1,0,1,0,1,0,1
1980 DATA 0,1,1,0,1,0,1
1990 DATA 1,1,1,0,1,0,1
2000 DATA 0,0,0,1,1,0,1
2010 DATA 1,0,0,1,1,0,1
2020 DATA 0,1,0,1,1,0,1
2030 DATA 1,1,0,1,1,0,1
2040 DATA 0,0,1,1,1,0,1
2050 DATA 1,0,1,1,1,0,1
2060 DATA 0,1,1,1,1,0,1
2070 DATA 1,1,1,1,1,0,1
2080 DATA 0,0,0,0,0,1,1
2090 DATA 1,0,0,0,0,1,1
2100 DATA 0,1,0,0,0,1,1
2110 DATA 1,1,0,0,0,1,1
2120 DATA 0,0,1,0,0,1,1
2130 DATA 1,0,1,0,0,1,1
2140 DATA 0,1,1,0,0,1,1
2150 DATA 1,1,1,0,0,1,1
2160 DATA 0,0,0,1,0,1,1
2170 DATA 1,0,0,1,0,1,1
2180 DATA 0,1,0,1,0,1,1
2190 DATA 1,1,0,1,0,1,1
2200 DATA 0,0,1,1,0,1,1
2210 DATA 1,0,1,1,0,1,1
2220 DATA 0,1,1,1,0,1,1
2230 DATA 1,1,1,1,0,1,1
2240 DATA 0,0,0,0,1,1,1
2250 DATA 1,0,0,0,1,1,1
2260 DATA 0,1,0,0,1,1,1
2270 DATA 1,1,0,0,1,1,1
2280 DATA 0,0,1,0,1,1,1
2290 DATA 1,0,1,0,1,1,1
2300 DATA 0,1,1,0,1,1,1
2310 DATA 1,1,1,0,1,1,1
2320 DATA 0,0,0,1,1,1,1
2330 DATA 1,0,0,1,1,1,1
2340 DATA 0,1,0,1,1,1,1
2350 DATA 1,1,0,1,1,1,1
2360 DATA 0,0,1,1,1,1,1
2370 DATA 1,0,1,1,1,1,1
2380 DATA 0,1,1,1,1,1,1
2390 DATA 1,1,1,1,1,1,1
99999 END
```

*Program from General Electric BASIC Mark II.

TABLE V

Explanation of Computer Program DISSEV

Program Lines	Function Performed
100-130	Input data described below.
140-150	Output file for sickness scores.
160-170	Storage allocation in computer.
180	Number of symptoms and precision of probability calculations.
190-220	Matrix size allocations.
230-290	Constants for probability calculations.
300-650	Computes probabilities for all L symptoms at 101 levels of the sickness score.
660-690	Writes these scores to a file.
730-760	Reads one symptom pattern from data lines 1130-2390 (7 values of 0 or 1).
770-840	Chooses probability of symptom presence or absence depending on occurrence or non-occurrence in pattern.
850	Forms product of probabilities so chosen.
890-960	Chooses and prints the sickness score yielding the largest probability product at line 850.
990	Reads another pattern until lines 1130-2390 are completed.
1000-1110	Computes probability integral.
1130-2390	All possible patterns of seven symptoms.

DIGITAL SIMULATION OF EQUIPMENT ALLOCATION FOR CORPS OF ENGINEER CONSTRUCTION PLANNING

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U. S. Army Corps of Engineers
Champaign, Illinois

ABSTRACT. Procedures for simulating resource requirements and efficient work organization for typical Corps of Engineers construction processes are developed. A user-oriented computer simulation language was adapted by providing guidelines and specifying procedures aimed at the construction planner and facility engineer. Network elements from GERTS are combined to build "modules" which frequently recur in the modeling of construction operations and processes. Representative modules are developed and their utilization is illustrated by modeling a typical earthmoving operation.

HISTORICAL PERSPECTIVE. Were it possible for the modern day structural engineer to talk with the master builders of the great European cathedrals, he might well be placed in the same situation that the modern day construction engineer finds himself vis-a-vis the construction manager. The structural designer's observations concerning the more "optimal" design of a flying butress might be listened to as interesting theory, but certainly not considered convincing enough to replace the master's intuitive feeling for the proportion of a given column or arch. Construction managers who have been successful using methods based on "engineering experience" react in the same way to new "theoretical" planning methods proposed by construction engineers. They will not relinquish proven intuitive approaches in favor of proposed methods until (a) the nature of the new methods are understandable to them, (b) the superiority of the methods are demonstrated in terms of profit dollars, and (c) the limitations involved are adequately explored. For these and other reasons the application of analytic methods to the design of construction operations has been extremely limited.

Operations Design in the Construction Industry

The design of operations has been an area of interest and research to engineers since F. W. Taylor first conducted his historic time and motion studies on the productivity of workers loading slag early in this century. Industrial operations have been prime candidates for study since in many cases they are of a repetitious nature and not normally subject to greatly varying environmental conditions. The techniques of operations research have proven excellent tools in designing industrial assembly lines, analyzing inventory policies, scheduling work patterns and the like.

Within the past decade, studies have been conducted in an attempt to apply the concepts of operations research to the area of construction management and the design of construction operations. These studies can be conveniently divided into two categories:

(1) Studies based on mathematical techniques such as queueing theory and linear programming (17,25,26)

(2) Simulation studies.

Construction Modeling Techniques

The preponderance of work done in the area of mathematical modeling has been concentrated on the application of queueing concepts to the solution of construction problems which are easily formulated within the context of single and multi-server models. The problem illustrated in Figure 1 is typical of the class of problems which have been attacked using queueing methods. The figure represents the so called "shovel-truck" problem. Given the arrival and service distributions of the trucks and shovel respectively, the problem is to determine the productivity of the system and the implications of varying the number of trucks as well as the $\frac{\lambda}{\mu}$ ratio, where:

λ = the arrival rate (poisson distribution)

μ = the server ratio (exponential distribution)

This problem and extensions of it have been considered in studies conducted at the University of Illinois (17,25,26). The models investigated, however, are subject to assumptions (e.g. steady state) which seriously limit their validity. Further, as the components and complexity of the system increase the calculational difficulties involved in solving the associated differential equations render the method intractable.

Simulation methods have been used notably by Teicholz in the selection of a heavy equipment fleet in support of large earthmoving operations (29) and by Arrington in the optimal dispatching of ready-mix concrete trucks from a central concrete batching facility. (1) However, no general approach to the simulation of a large and complex construction operation has been developed to the authors' knowledge.

Shortcomings of Present Techniques

The actual application of both the analytic and simulation concepts studied has received very limited documentation in the literature. This indicates that penetration of the industry by these methods has been marginal. Several reasons for this lack of application are apparent.

(1) Construction projects, although they contain a certain set of common processes, tend to be unique in character. Repetitious work activities which lend themselves to analysis are common to the micro-level processes of construction (e.g. brick-laying, concrete placement, etc.)

However, as the processes are synthesized at the macro-activity level (e.g. place concrete for 1st floor slab) site and project peculiarities lend a uniqueness to the situation. That is, the synthesis of micro-processes into macro-activities requires a special treatment or method of arrangement which is characteristic of the project at hand.

(2) Construction operations are subject to a constantly changing environment. The transient nature of the operational environment and the parameters associated with it tend to invalidate the results of conventional analysis before they can be applied.

(3) Sophisticated methods requiring highly trained personnel and expensive support (i.e. computer time, etc.) have not shown themselves efficient and general enough in giving timely answers which can convince profit-oriented construction managers that they are an improvement over "rules of thumb" and the intuitive methods which are in use today.

A NETWORK SIMULATION APPROACH TO CONSTRUCTION OPERATION DESIGN. Queueing type network simulations have been proposed as a means of attacking a wide range of large scale systems problems (6,16,18,19). This method of modeling allows the analysis of complex systems characterized by a high degree of interaction between systems components. The speed of computer simulation methods and increasingly accessible and usuable simulation languages provide the basis for quick problem formulation and analysis. It appears feasible using simulation to reduce problem processing time to the point that timely solutions can be achieved.

The improvements in problem formulation and solution time using network simulation accrue because:

(1) Simulation eliminates the requirement of formulating the differential equation space and avoids the necessity of explicitly solving the state equations. It allows direct access to the statistical information of interest.

(2) Simulation formulation of a given productive system lends itself to a modularization which greatly simplifies the modeling process. Certain components recur so often as to constitute modeling blocks which can be arranged much in the same fashion one arranges the modules on an analogue "patch" board. These modeling blocks provide the requisite versatility required to structure a wide range of practical operating systems.

(3) Simulation also allows investigation of transient performance and statistical fluctuation analysis as well as network structural analysis such as vulnerability and sensitivity analysis. These types of investigation are not possible using mathematical state models since the solution of the differential equations involved imposes steady state restrictions on system operation.

Scope of GERTS Q Simulation

The GERTS III Q Simulation program exploits the speed and ease of solution possible with digital simulation of a queueing network. It further provides a new and improved approach to problem formulation, responsive to operational engineering requirements.

GERTS is an acronym for Graphical Evaluation and Review Techniques Simulation and was developed under a five-year old cooperative project of about twenty operations research teams at universities, in government agencies and industry under the leadership of Allan Pritsker of Purdue University.

The development and utilization of GERTS is documented in several hundred internal reports, which are condensed and brought into the public domain mostly through tutorial papers:

Concepts and Theory: (11, 19, 21, 22, 32)
Program Development and Diagnostic: (18, 20)
Model Development and Use: (5, 30, 31)
Application to Industrial Systems: (27)
Application to Social Systems: (2, 7, 24)

Users of GERTS/Q exceed 200 including about 20 universities, usually within the framework of courses on operations research. User-oriented simulation programs, tested and diagnosed by these groups, are directed towards (a) cost allocation: GERTS III C, (b) resource allocation: GERTS III R, and (c) queueing networks: GERTS III Q.

Only the utilization of GERTS III Q is reported here based on the day-to-day needs by a designer of construction operations. Operating and coding instructions are well documented and are briefly surveyed merely to make this presentation self-contained. The primary aim of the procedures to be developed is to be useful to engineers, faced with the task of modeling construction systems of terms of stochastic processes. The GERTS III Q program models the system of interest in terms of a network representation. The network is presented using arrow notation. That is, arrows are used to represent activities and nodes represent event markers.

A tabular presentation of the elements used in modeling queueing networks with GERTS is given in Table 1 to Table 3.

GERTS III Q Modeling Blocks

GERTS III Q is particularly flexible in the development of modules or building blocks which can be used in the modeling of construction operations. Three modules constructed using GERTS are described in both simplified and GERTS III Q network format in the following sections.

These three sub-systems are designated based upon their function as:

(1) Pairing Module
(2) Distributor Module
(3) Selector Module

They demonstrate the procedure involved in developing models for practical requirements and constitute three representative entries typical of a file or library of compatible modules. Drawing upon this resource file, complex systems such as interactive construction processes can be modeled by repetitively using the modules in various combinations. The advantage of this concept is that sub-networks can be validated against observed phenomena and then combined with other validated sub-networks to build a process model. This model can, in turn, be re-validated and adjusted to account for secondary and interactive effects resulting from the combination. Use of such a complex model allows investigation of varying resource assignment schemes and their effect on the overall productivity of the construction process. By this means the manager is in a position to locate "bottlenecks" and ascertain critical factors before committing to a particular scheme of execution.

REPRESENTATIVE MODULAR STRUCTURE.

GERTS Structure for Pairing Module:

The pairing module is encountered in modeling many construction operations. Table 4 indicates several typical operations for which the pairing module is required. Table 5 gives a detailed description of the P-Module (Pairing Module) and its characteristics.

The pairing module can be developed from the basic elements of GERTS by synthesizing three interacting mechanisms.

a. Pairing Mechanisms:

Channel 21-24-41 represents the service demanded path (MN) in the state network. Channel 42-41 maps to the server path AB. In the truck-loader situation, the truck requiring service enters along 21-24-41. At this point in the absence of a loader it waits. The loader enters initially along path 40-41 and subsequently along 42-41. In the absence of a truck it waits since 41 requires two entities to be realized. When both a truck and a loader are present the pairing requirement is satisfied and the service procedes (41-42-43).

b. Filtering Mechanisms:

To preclude the situation where 2 trucks arriving along 21-24-41 cause realization of 41 and initiation of the service without the pairing requirements being satisfied, a filtering structure is used.

This structure is based upon the branch modification feature available in GERTS. To implement this operation 24-41 is designated activity 1 and 42-43 activity 2. Upon consummation of activity 1 entity flow is rerouted from 23-24 to 23-25. The loop (23-25-26), constitutes a holding pattern which captures and retains entities which must await pairing.

c. Switching Mechanisms:

To synchronize the pairing and the filtering operations a switching mechanism is required. The switching mechanism provides for the routing of entities to the holding pattern when appropriate. Further, it re-establishes the upper pairing branch (e.g. 23-24-41), when a service has been completed. The switching scheme is also shown in Table 5.

GERTS Structure for Distributor Module:

The purpose of this module is to provide the capability of routing incoming entities to servers in an order of ranking. It is used in modeling when a preference scheme among processors is to be used. The distributor module is synthesized from two mechanisms: (a) switching mechanism, and (b) a holding mechanism. Table 6 provides a detailed description of the D-module (Distributor Module) and its properties.

a. Switching Mechanism:

Channel 21-23-24 (see Table 6) represents the initial entry channel to the distributor module. Again using a truck as the entity to be served, let us postulate the situation that the truck has three entry points to a construction yard. The truck checks the gates in a given order of preference. The order of preference is gate 1 first, gate 2 and gate 3 (in that sequence). In Table 6, processor channel 24-25 representing gate 1 is first checked for availability. If busy, (i.e. if another entity is in this service state) path 23-24 will be closed and the incoming truck will be diverted via 23-33 to check the second gate or service state. The closing of path 23-24 once an entity has entered service state 24-25 is affected by the switching scheme also given in Table 6. Once an entity has entered the service state (24-25) by traversing 23-24, the network modification characteristic of GERTS III Q allows switching of subsequent entities to node 33 via 23-33. When the service is completed, the served entity (truck) exits the service state (gate) via 25-26. The traverse of 25-26 causes subsequent entity flow to be re-established through 23-24, since completion of 25-26 results in a re-constitution of this path.

Similarly a check is made of service states 2 (34-35) and 3 (44-45) to see if they are available. If so, the truck enters the service state (gate). If not, flow is diverted as described above.

b. Holding Mechanism:

If all three gates are occupied (all servive states busy) then the entity is diverted to Q node 50. The processing time required in activity 50-22 will establish a holding loop 22-23-33-43-50. Upon availability of one of the occupied service states, any entity in the holding pattern will exit via 24, 34, or 44. Should more than one entity be in the holding loop, one or more entities will be at Q node 50 and another will be in transit via 50-22. Should only one entity be in the holding mechanism, it will be in service activity 50-22. Should this occur it will not be available for re-entry to the system until it has completed this "dummy" service. The service time T for 50-22 should be selected to establish a meaningful holding time. It may cause a slight deviation from "optimal" processing, but in most operational processes a certain pause or delay is typical in moving from holding to service. The service time for 50-22 should be selected to be representative of this delay.

GERTS Structure for Selector Module:

The selector module is encountered when processing entities of varying priorities. Its purpose is to give preferential service to higher ranking entities. For instance, if two truck types are being used and one type is to be given preference, this module is applicable. Table 7 gives a detailed description of the S-Module (Selector Module).

This module is also based upon three interacting mechanisms.

a. Holding Mechanism:

Priority entities flow from 21 to 42 if the service state is free. If the service state is occupied, it is implicit that activity 2 has been traversed and following priority entities are diverted to holding loop 23-51-50. Similarly, 2 causes diversion of routine entities to holding loop 33-60-61. When an entity exits the service state (40-41), activity 3 (41-42) is traversed and releases entities from the holding mechanism.

b. Biasing Mechanism:

The purpose of the biasing mechanism is to hold one entity in order to give preference to a higher ranking entity. This operation constitutes a biasing. This is accomplished in Table 7 by activity 2 triggering the path modification 34-60. Due to the time delay, t_d, associated with activity 33-34, the higher ranking entity advances through 24 to 40 triggering the above modification, while the low ranking entity is delayed in 33-34. The modification causes the low ranking entity to be diverted back to 60 and into the holding mechanism.

c. Switching Mechanism:

The switching scheme is shown in Table 10. The priority entity upon passing through activity 1 (22-23) diverts incoming routine entities into the routine holding loop (33-60-61) by modifying path 33-34 to 33-60. The consummation of activity 2 activates paths 23-51 and 33-60 which divert entities to the holding mechanism. Additionally, 2 removes path 34-40 and replaces it with 34-60. This is to implement the operation of the biasing mechanism as explained above. Upon completion of the service, the entity exits via 3 and causes resetting of paths 23-24 and 33-34.

APPLICATION TO EARTH MOVING SYSTEM

Model Formulation for Earth Moving Process

The earth moving problem is continually encountered in both horizontal and vertical construction; it represents a fundamental and recurring construction process. The problem examined in this section consists of several activities typical of an earth moving process:

a. a bulldozer pushes material into a stockpile,
b. two loaders load material into four trucks,
c. the trucks haul the material into a fill location,
d. the truck finally returns to reload.

Fig. 2 describes this process as a state diagram. Fig. 3 models the same process in terms of the modular concept developed in the previous section.

Tables 8 and 9 describe in detail a GERTS network for each module used in Fig. 3. The entry and exit points for each module correspond to the interconnection of different modules in Fig. 3. These tables also interpret the time sequence of entity flow through each module. Table 10 summarizes the relationships between the states of Fig. 2 and the modules of Fig. 3. The switching schemes where appropriate are given in Table 11.

Construction of State Diagram for Truck Loader Problem

The complete earth moving process is characterized by seven states, which correspond to "activities." One of these is the waiting status of the processor.

State 1: The bulldozer pushes material into a stockpile position.
State 2: The bulldozer returns to a start position preparatory to pushing up more material.
State 3: The material pushed up by the bulldozer resides in its stockpile position.
State 4: These entities combine in this state to consummate the loading operation, which requires the presence of a truck, a loader and a truckload of material.

State 5: The loader, lacking either a truck or a load of material, returns to its waiting state.

State 6: The truck with its load of material travels to the fill location, dumps and returns for reloading.

State 7: The truck awaits the arrival of a material load and the availability of a loader.

This state presentation has the following important characteristics:

a. It constitutes the initial level of hierarchy of modeling procedures and is based on observations.

b. To identify significant states, one must have facility in formulating a problem in terms of activities (active and waiting).

c. The set of states provides the input to a modular representation.

Modular Representation

Fig. 3 describes the earth moving problem in terms of six basic module types. The functions of the modules and their relationships to the states of Fig. 2 are as follows:

G-Module: Generates the cycle time of the bulldozer and handles the functions performed in State 1 and State 2 of the state diagram.

D(1)-Module: Performs the function of distributing the truck loads of material generated by the bulldozer to the loader positions. It represents essentially State 3 in the state diagram.

P-Modules: Two identical modules perform the function of combining (pairing) the material the truck and the loader in the loading operation. They also generate the time duration of the loading process. They represent the functions performed in State 4 and State 5.

D(2)-Module: Provides an overflow (or holding) location for trucks awaiting loading. It represents State 7.

I-Module: Performs the function of initially entering the four trucks of the model into the system. This module has no counterpart in the state diagram.

A-Modules: Generate the travel-dump-travel time associated with the trucks. They perform the function required in State 6.

These modules map closely into the states of the state diagram, but the two representations differ and are complementary in these aspects:

a. the nature of the state diagram is essentially activity based.

b. the nature of the modular representation is based on entity operation.

c. the mapping is not exact whenever activity and entity operation differ since associated with each state are varying numbers of entities

d. going from the state diagram to the modular representation a quantitative formulation has to be generated from a qualitative formulation. For example, handling or processing functions are implicit in the state diagram, but have to be considered explicitly in the modular representation as it occurs for the distribution function. The relationships between modules and states are summarized in Table 10.

Modular Networks for the Flow of Entities

The networks corresponding to the eight modules used in Fig. 3 are shown in Tables 8 and 9.

G-Module: The process of earth moving is initiated at node 2, which generates random time intervals for the production of material entities or pulses. This is achieved by the looping activity 2-2, which has associated with it the cycle time of the bulldozer.

D(1)-Module: Material is distributed by this module from the G-module into the P-module. It also provides a storage capability (Q-node 15) should the P-modules be occupied. The operation is the same as the distributor module described in section 3, except for the number of exits.

D(2)-Module: The function of the D(2)-module is the same as the D(1) except it distributes trucks to the P-modules. It also can store trucks (Q-node 45) should the P-modules be occupied.

P-Modules: The pairing modules in the process model differ from the pairing module in section 3 only in complexity. In this case both P-modules combine three entities instead of two, resulting in two input ports instead of one. Additionally, it will be noted that in this configuration with the D-modules the filtering mechanisms (see above) are not required. The filtering effect is achieved by the D-modules. The entities combined are the loader, the truck and the truck loader material.

I-Module: To initialize the truck flow into the system, the I-module provides input from four source nodes (51,52,53 and 54). The four source nodes provide the initializing entity flow for the trucks in the earth moving operation.

A-Module: The A-module simply generates the activity time which is used to simulate the travel-dump-travel time of each truck on the haul. This is achieved using activities 61-62 and 71-72.

The validation and testing of each network model allows clearer insight into the operation of individual components of the earth moving system. Additionally, just as the modular concept is of advantage in structuring a problem in terms of recurring functions and components, this procedure offers a systematic approach to testing and failure diagnostics:

a. Each module can be accessed separately and validated for fidelity and accuracy.

b. The interconnection between modules can be diagnosed for problems generated at the interface of two modules.

c. The propagation of error throughout a system can be assessed in terms of contribution of each module.

d. Finally the vulnerability of a systems in terms of vulnerability and redunancy of specific modules can be analyzed.

Entity Time Sequence in Modules

Column III in Table 9 describes the entity flow through the complex modules by plotting the flow at each node versus time.

A time line (illustrated using bars) is shown for each significant node or activity in the network. The presence of an entity in the node N at time increment T, is shown by a letter representing that entity (e.g. "A""E""F") on time line N at the location representing time T, where N is the node or activity of interest.

In this example, the letters E, F, G and H refer to (quanta of) earth material, T to Trucks, L to Loaders, A to the presence of both E (or F, or G, or H) and L, B to the presence of E and T and finally C to presence of E (or F, or G, or H) together with L and T (e.g. E, L and T).

Rules for Time Sequence Diagrams

In the interpretation of the time sequence diagram, the following rules are to observed:

1. Entities: The presence of an entity at a node or in an activity is given by a letter representing that entity on the time line representing the node.

2. Direction of Flow: Arrows between time lines indicate movement between nodes or into activities.

3. Interval of Occupancy: The position of the letter along the node timeline indicates the time interval of occupancy.

4. Duration of Activity: The movement between nodes and through activities is governed by the type of node, the time delays associated with activities, and the switching mechanisms used.

5. Barred Sequences: The bar formed by the sequence of letters reflects the time an entity spends in a given node or activity.

6. Time Lines: The letter N at the beginning of a time line represents the node associated with the time line.

7. Super Nodes: The symbol NM denotes that nodes M and N are considered together as a supernode and that the time line is associated with both. That is, the presence of a letter on time line NM means the entity is either in N, in M, or between.

CONCLUSION. The method presented provides a logical vehicle for describing and examining large and complex construction operations. By developing modules of small and recurring modeling functions and combining these in an interconnecting system of arbitrary complexity, it is possible to analyze construction operations in far greater detail than possible by existing methods.

In order to fully implement the potential of this method, work is in progress to achieve the following:

a. Systematic development of additional modules based on a wide range of examples from Corps of Engineering practice.

b. Clearer insight into the hierarchy of mechanisms, modules and fundamental operations to form ultimately a compact and efficient data processing scheme.

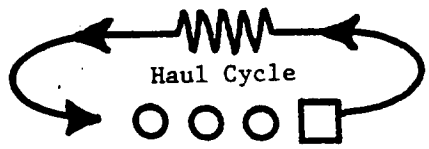
c. A user interface, facilitating user input by the practicing engineer in terms of a problem oriented language.

d. Assessment of limitations of methods.

e. Validation of models against observed operation.

The network approach made possible by GERTS III Q permits us to formulate our problem in a method most convenient and economical to the computer. The value of network simulation is in establishing a link between data processing by computer and the examination of earthmoving operations in a conventional way. The more effectively these two points of view can be reconciled the more effective will be the utility of this methodology in modeling practical engineering situations.

secondary operation (i.e. dumping)

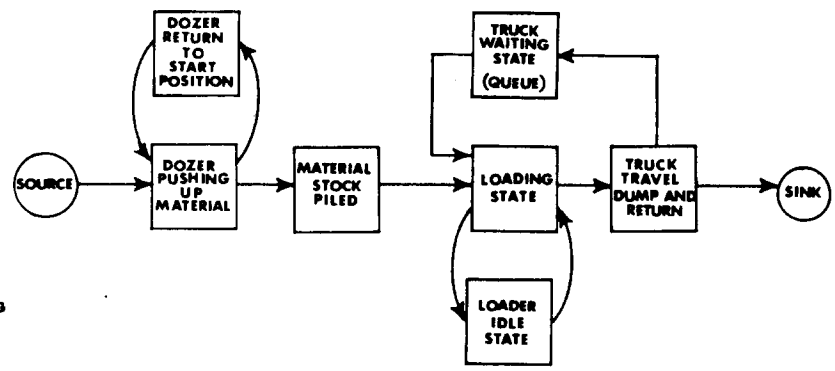


server = shovel or loader

trucks

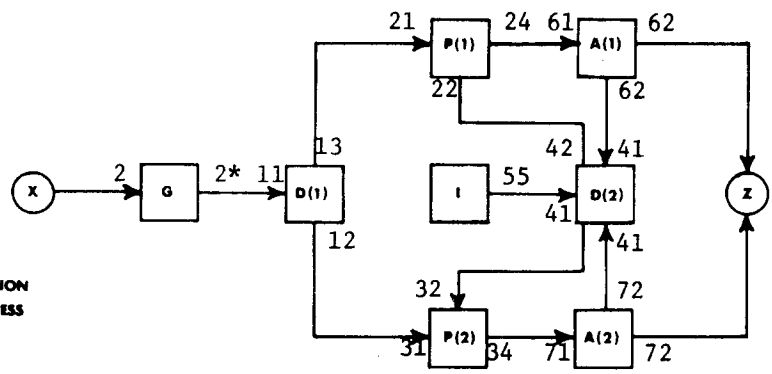
Single Server Model (State Model)

FIGURE 1



STATE DIAGRAM FOR PROCESS EARTHMOVING

Fig. 2



MODULAR REPRESENTATION OF EARTHMOVING PROCESS

Fig. 3

TABLE 1. PROCESS NODES

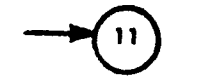
Source Node	
Sink Node	
Event Node	

TABLE 2. OPERATIONS NODES

Operation	Node	A Entities Needed	B Entities Needed
Event		For initial realization	For subsequent realization
Queue		Initially in Q	For Q Limit

TABLE 3. GENERATOR OPERATIONS

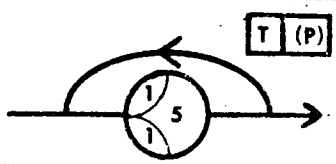
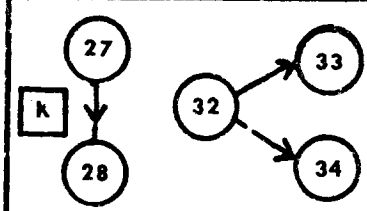
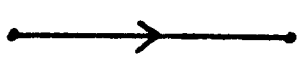
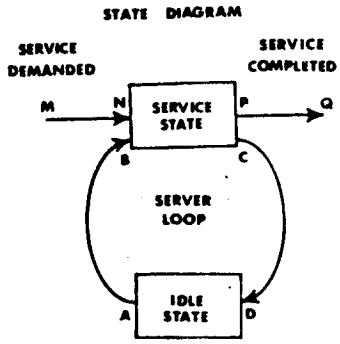
Generator Symbols	Functions Generated	Description of Function
	Entity Generator	This activity generates activities based on time T and probability distribution P
	Modification Generator	The trigger K has been activated by an entity passing from 27 to 28; as a result path 32-33 is replaced by path 32-34.
PARAMETERS 	Probabilistic Generators	Several parameters such as time delay, probability, and others are associated with this activity.

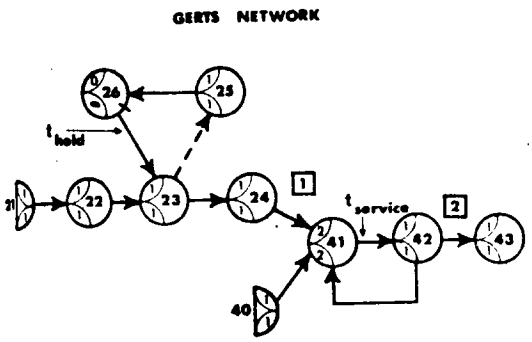
TABLE 4: Examples of Typical Operations

Service Demanded	Service State	Waiting State	Service Completed
Load Fill into an empty truck	Truck loaded by loader	No trucks to be loaded by loader	Filled truck ready to depart
Movement of a piece of steel	Lifting of steel by crane	No steel remaining to load	Steel in desired location
Movement of a brick pallet from grade to 2nd floor	Lifting of pallet by a construction lift	No pallets to be lifted	Pallets in desired location

TABLE 5: PAIRING MODULE



Description of Operation:
Entities arrive for service along the service demanded channel (MN). Upon their arrival the server moves from its idle state to the service state via path AB. Both the presence of the server and of an entity demanding service are pre-requisite to the initiation of a service. Upon the completed service, the served entity exits the service state via path PQ and the server returns to its idle state via CD.

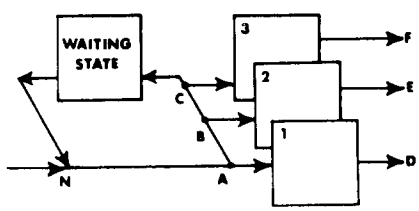


Switching Scheme:

Activity realized	1	3
Replace Node	24	25
With Node	25	24

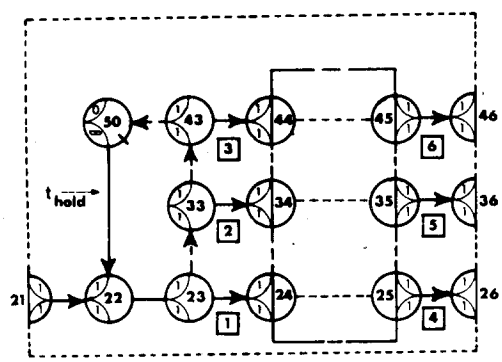
TABLE 6: DISTRIBUTOR MODULE

STATE DIAGRAM



Description of Operation:
Entities enter along NA. If processor 1 is available, it continues along route AD. If processor 1 is busy, the entities travel AB and check processor 2. If 2 is free, it continues along BE. If 2 is busy, it travels BC and checks processor 3. If all 3 are busy, it waits until one becomes free.

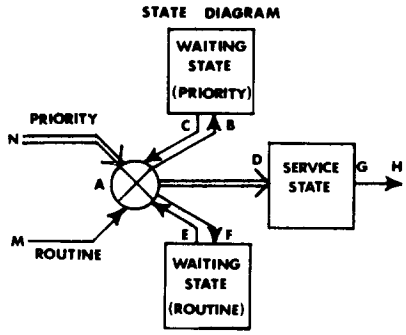
GERTS NETWORK



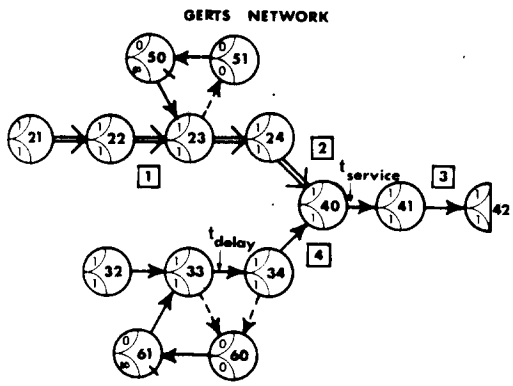
Switching Scheme:

Activity Realized	1	2	3	4	5	6
Replace Node	24	34	44	33	43	50
With Node	33	43	50	24	34	44

TABLE 7: SELECTOR MODULE



Description of Operation:
Priority entities enter along path NA. Routine (e.g., lower priority) entities enter along MA. If the service state is free, the entity proceeds through AD to that state. If the service state is occupied, the entity is diverted to a waiting state depending upon its type. When the service state is available, priority entities are processed until the priority waiting state is empty. When this situation obtains the routine entities are processed from their waiting states.



Switching Scheme:

Activity Realized	1	2	2	3	3	3	4
Replace Node	34	24	40	51	60	60	34
With Node	60	51	60	24	40	34	60

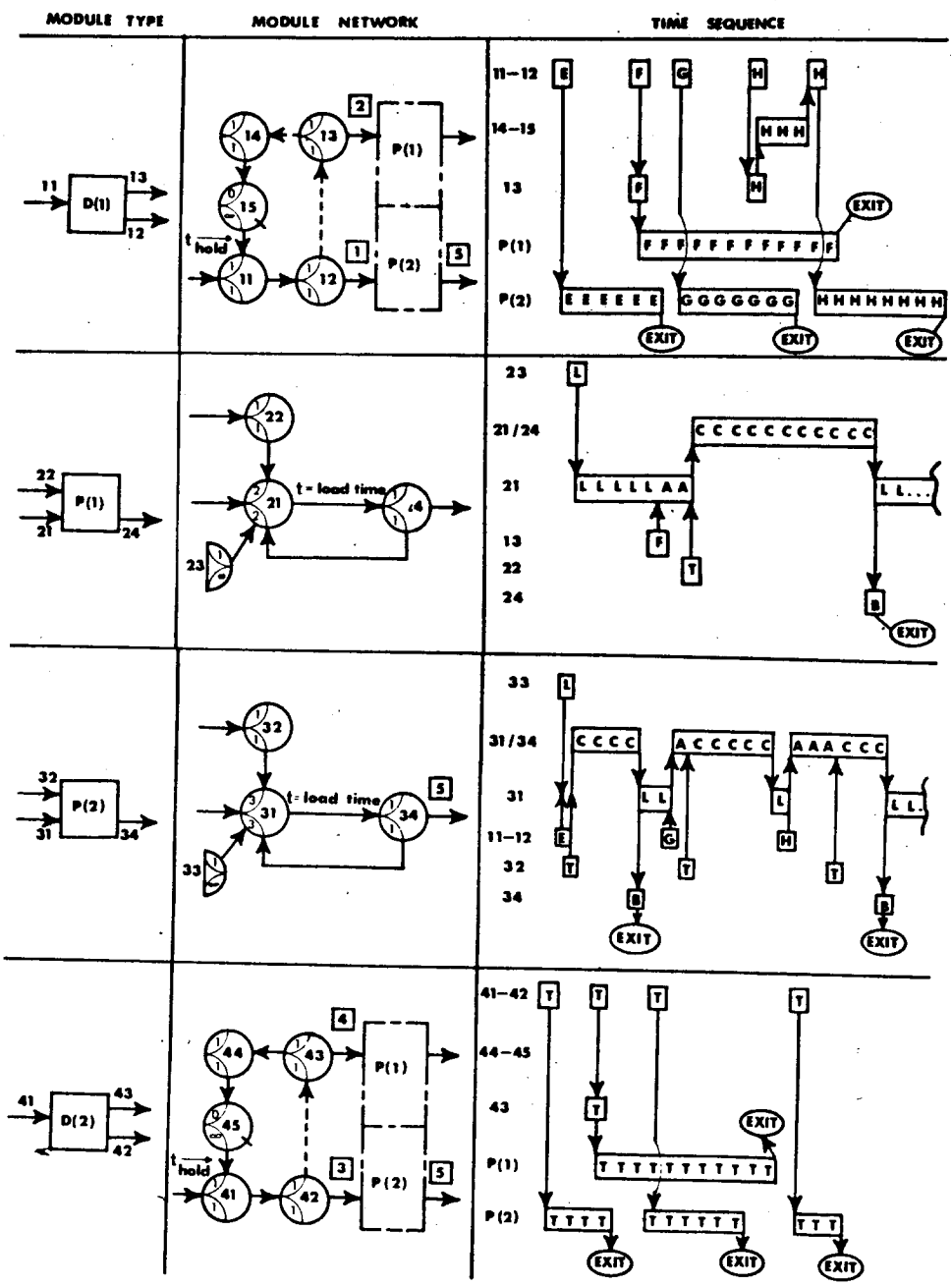


Table 9: Complex Modules

Table 8: Simple Modules

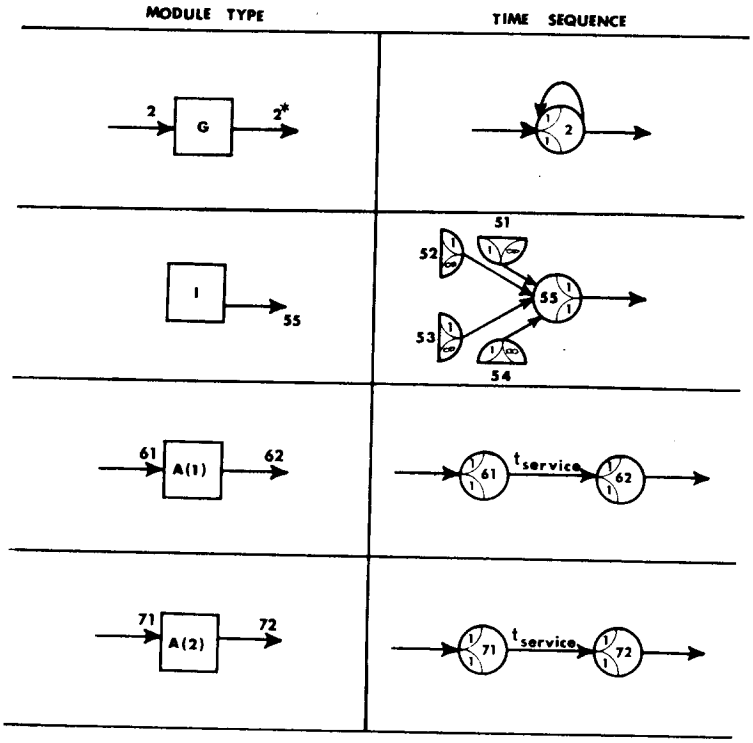


TABLE 10:

RELATIONSHIP BETWEEN

MODULES AND STATES

MODULE TYPE	STATE	BULLDOZER	LOADER	MATERIAL	TRUCK	FUNCTION PERFORMED
G	1,2	X		X		Generate Bulldozer Production
D(1)	3			X		Stockpile or Distribute Material to Loaders
D(2)	7				X	Hold or Distribute Trucks to Loaders
P(1) & P(2)	4,5		X	X	X	Combine Loader (L), Truck (T), and Material (E,F,G,H) for Loading
I	None				X	Initialize Four Trucks in
A	None			X	X	Generate Combined Travel & Dump Time

TABLE 11: EARTHMOVING PROCESS - SWITCHING SCHEME

Activity Realized	1	2	3	4	5	5
Replace Node	31	21	32	22	43	13
With Node	13	14	43	44	32	31

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MACHINE GUN EFFECTIVENESS MODEL BASED ON STOCHASTIC VARIATIONS OF THE BARREL DURING FIRINGS AS APPLIED TO HEMISPHERE TARGETS

Captain Richard H. Moushegian
Systems Research Division, R D & E Directorate
HQ., U. S. Army Weapons Command

INTRODUCTION:

1. Good Morning, Ladies and Gentlemen. I am Captain Richard Moushegian of the Systems Research Division at the US Army Weapons Command.

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M A C H I N E G U N E F F E C T I V E N E S S M O D E L

BASED ON STOCHASTIC VARIATIONS OF THE BARREL

DURING FIRINGS AS APPLIED TO

HEMISPHERE TARGETS

SLIDE #1

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2. PURPOSE: The purpose of this MG model is to study the effectiveness of MG fire on a point target of hemispherical configuration.

3. APPROACH: The approach used is to utilize Monte Carlo simulation routines to vary certain initial conditions at the muzzle and to observe the projectile's effect on a three-dimensional target region. On the next slide are some measures of target effectiveness which are used.

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MEASURES OF EFFECTIVENESS

1. P(F) - THE PROBABILITY THAT THE FIRST ROUND IN A BURST INTERSECTS WITH THE TARGET REGION.

2. P(S) - THE PROBABILITY THAT EACH SUBSEQUENT ROUND IN A BURST INTERSECTS WITH THE TARGET REGION.

3. E(H) - THE EXPECTED NUMBER OF ROUNDS IN A BURST WHICH SHOULD INTERSECT WITH THE TARGET REGION.

4. MEAN AND VARIANCES OF IMPACT POINTS ON THE TARGET SURFACE FOR FIRST ROUNDS AND SUBSEQUENT ROUNDS.

SLIDE #2

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4. PREVIOUS DEVELOPMENTS:

a. Some previous analytical work in MG effectiveness area as shown on the next slide was performed by Mr. Herbert Fallin of Army Materiel Systems Analysis Agency (AMSAA) in the document TM-33.

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PREVIOUS DEVELOPMENTS IN MG EFFECTIVENESS AREA:

1. AMSAA DOCUMENT TM-33 BY MR. HERBERT FALLIN

2. USAWECOM REPORT SY-R3-71 BY DRS. T. H. HUNG AND J. T. WONG

SLIDE #3

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His work was based on a two-dimensional target, and some assumptions made wrt displacement of first round from subsequent rounds at the target.

b. Another approach to the MG effectiveness problem was performed by Drs. Hung and Wong of the Army Weapons Command in SY-R3-70. Their work was based on three-dimensional targets of different configurations and some assumptions were also made wrt displacement of the first round from subsequent rounds at the target.

c. SCOPE: However, the scope of the following MG effectiveness study is based on a particular three-dimensional target with several assumptions being made at the weapon system rather than at the target. The three-dimensional target was a hemisphere only because at the time our office was asked to develop a hemisphere model for an on-going project at the Weapons Command.

d. For the remaining portion of this presentation, I will use the outline noted in slide 4.

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O U T L I N E

I. THREE (3) BASIC ASSUMPTIONS

II. BASIC FORMULAS

III. COMPUTER PROGRAM DISCUSSION

IV. NUMERICAL EXAMPLE

V. CONCLUSIONS

SLIDE #4

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Prior to continuing, I would like to point out one important operational characteristic of MG weapons systems. This characteristic is that it has been observed that during automatic fire the first round of a burst hits the target in one location while the remaining rounds of the same burst seem to be nested in some other location significantly separate from the first round. This separation is primarily due to a weapon-mount interaction problem where the weapon is already in motion when the subsequent rounds are fired.

* * * * * * * * * *

I. THREE (3) BASIC ASSUMPTIONS:

A. INITIAL POSITION FOR FIRST ROUND OF A BURST IS SIGNIFICANTLY DISPLACED FROM THE CENTROID OF INITIAL POSITIONS FOR SUBSEQUENT ROUNDS.

SLIDE #5

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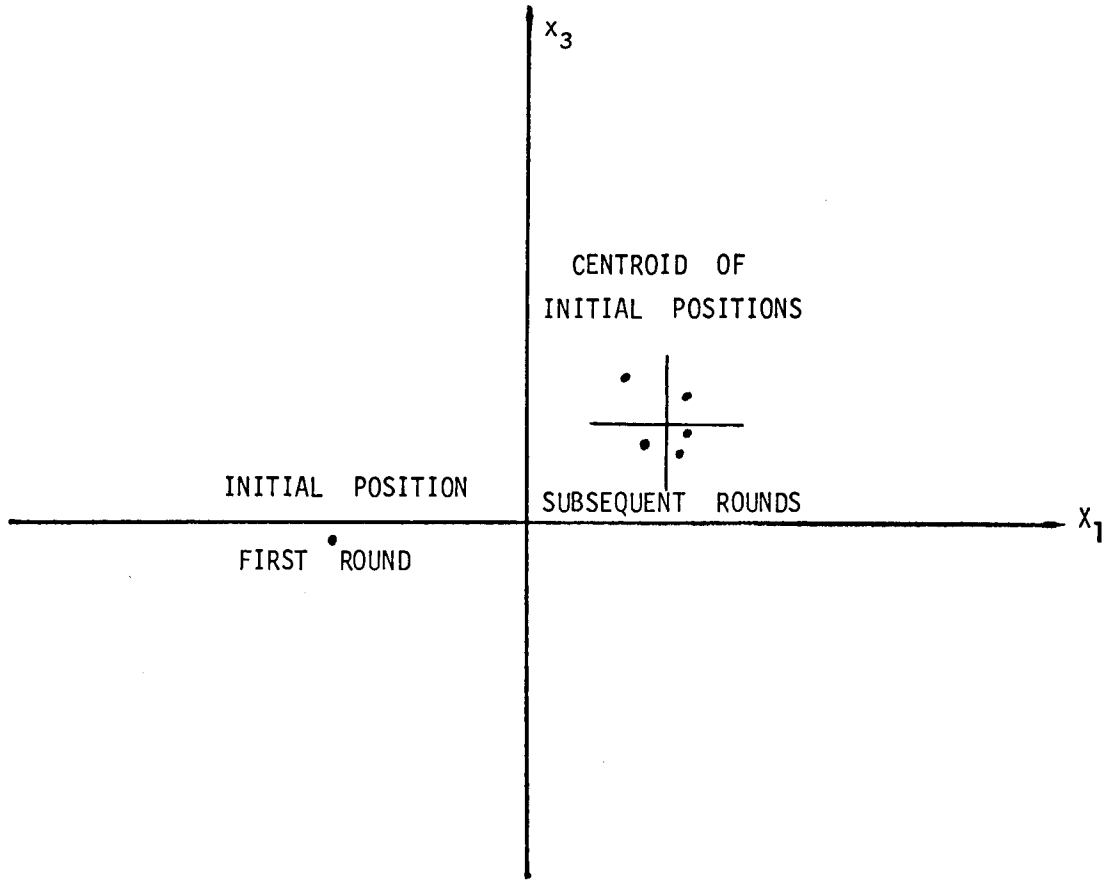
(1) Three (3) Basic Assumptions:

(a) In regard to the assumptions of this study, the first two are concerned with initial projectile flight conditions just as the projectile is launched from a MG barrel which is experiencing stochastic excitations during automatic fire.

(b) Specifically, the first assumption is that the initial position of the first round of a burst is significantly displaced from the CENTROID OF INITIAL POSITIONS for subsequent rounds. The next slide illustrates this significant displacement (or offset) as it applies to this modeling effort. This offset was probably first documented by AMSAA in TM-33. In that report, the displacement was noted at the target. However, in this study, again the assumption is imposed at the muzzle during projectile launching.



SIGNIFICANT DISPLACEMENT OF FIRST ROUND FROM
CENTROID OF INITIAL POSITIONS FOR SUBSEQUENT ROUNDS

SLIDE #6

(c) The second assumption (on the next slide) is that the initial conditions of the projectiles at the muzzle were random such that five trivariate normal distributions characterize the initial dispersion of the rounds for flight time to t_0.

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A S S U M P T I O N S:

B. THE INITIAL CONDITIONS OF THE ROUNDS AT MUZZLE ARE RANDOM SUCH THAT FIVE (5) TRIVARIATE DISTRIBUTIONS ARE USED. A DISTRIBUTION IS USED FOR

(1) INITIAL POSITION FOR FIRST ROUND

(2) INITIAL VELOCITY FOR FIRST ROUND

(3) INITIAL POSITION FOR SUBSEQUENT ROUNDS

(4) INITIAL VELOCITY FOR SUBSEQUENT ROUNDS

(5) OFFSET DISTANCE BETWEEN INITIAL POSITION FOR FIRST ROUND OF A BURST AND THE CENTROID OF INITIAL POSITIONS FOR SUBSEQUENT ROUNDS OF A BURST.

C. ANY GIVEN ROUND FOLLOWS A DESCENDING PATH RELATIVE TO A HORIZONTAL PLANE AS IT PASSES THROUGH OR BY THE TARGET.

SLIDE #7

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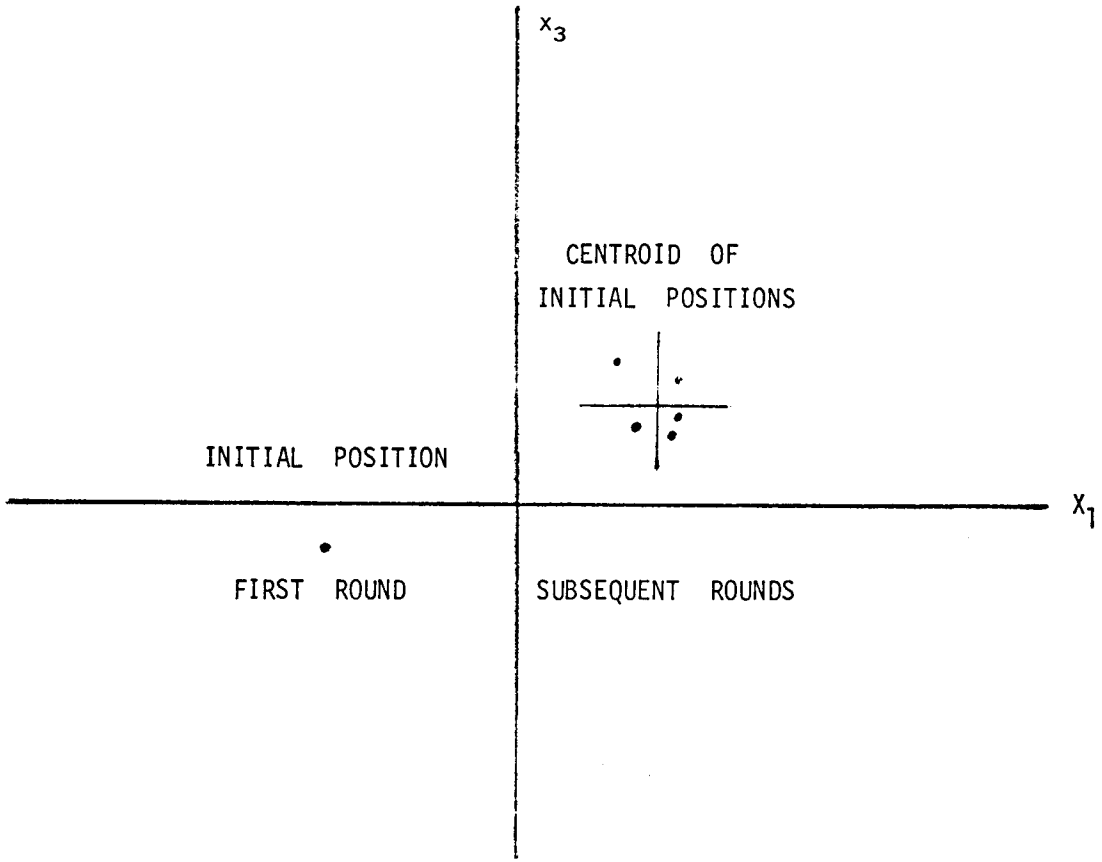
The three arguments per distribution are used to characterize performance in Euclidian three-space. Hence one trivariate normal distribution is used to characterize each of the following for t_0:

(1) Initial position for first round.
(2) Initial velocity for first round.
(3) Initial position for subsequent rounds.
(4) Initial velocity for subsequent rounds.
(5) Offset distance between initial position for first round of a burst and the CENTROID OF INITIAL POSITIONS for subsequent rounds of a burst.



[OFFSET: $\underline{x_{i30}} = \mu_{x_{i20}} - \underline{x_{i10}}$]

SLIDE #9

(d) The third and final assumption is that any given round must follow a descending path relative to a horizontal plane as it passes through or by the target. Note that this requirement does not necessarily require that the weapon system be located on or above the plane of the target. It only requires that the round be descending when it is in the general vicinity of the target region. This assumption is made for computational convenience and should not materially alter the results.

(2) Basic Formulas

(a) On the next slide the first Basic Formula is for offset distance.

* * * * * * * * * *

II. BASIC FORMULAS

A. OFFSET DISTANCE:

$$x_{i30} = \mu_{x_{i20}} - x_{i10} \quad \text{for } i = 1,2,3$$

SLIDE #8

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(1) The equation shown here shows offset distance as the difference between centroid of initial positions for subsequent rounds and the first-round initial position. (The i index denotes three-space.)

(2) In practice, a first round initial position X_{i10} is sampled from its trivariate normal distribution. (For notation purposes, the "1" subscript in the equation refers to the fact that it is a first round, and the "0" refers to the time parameter.) After the first round, an offset distance X_{i30} is chosen from its normal distribution. (The "3" index indicates offset and the "0" is still the time parameter.) Together with those two pieces of data, a CENTROID OF INITIAL POSITIONS [See next slide #9] for subsequent rounds $\mu_{X_{i20}}$ is then determined.

(The "2" index indicates subsequent rounds.) Now with a centroid (or mean), the subsequent round distribution has a location dependent upon the first round, and with a given set of variances for three-space, the subsequent rounds may be obtained without further delay.

(b) The basic FLIGHT EQUATIONS used in this effectiveness study are shown on the next slide. The first three equations give the projectile velocity for all three directions for any time t; and the second three equations give the projectile position for any time t. The first term after the equality mark for each equation is either initial velocity or position component. The g is gravitational acceleration constant, and the γ is related to projectile drag. This drag is a function of projectile mass and caliber, atmospheric density and velocity of sound.

(c) The WEAPON EFFECTIVENESS equations are very simply determined since the Monte Carlo approach was used.

(1) The $P(F) = \frac{\text{No. of target hits per simulation}}{\text{No. of rounds per simulation}}$

(2) The P(S) is basically the same.

(3) The E(H) is the equation shown on screen. Since our definition of P(S) was the prob. of each subsequent round hitting the tgt, we are allowed to multiply by the proper number of subsequent rounds per burst.

B. FLIGHT EQUATIONS:

$$v_{1jt} = v_{1j0}\, e^{\gamma t}$$

$$v_{2jt} = v_{2j0}\, e^{\gamma t}$$

$$v_{3jt} = v_{3j0}\, e^{\gamma t} + \frac{g}{\gamma}\,(1 - e^{\gamma t})$$

$$x_{1jt} = x_{1j0} - \frac{v_{1jt}}{\gamma}(e^{-\gamma t} - 1)$$

$$x_{2jt} = x_{2j0} - \frac{v_{2jt}}{\gamma}(e^{-\gamma t} - 1)$$

$$x_{3jt} = x_{3j0} - \frac{v_{3jt}}{\gamma}(e^{-\gamma t} - 1) + \frac{g}{\gamma}\left(t + \frac{e^{-\gamma t} - 1}{\gamma}\right)$$

for $j = 1, 2$

$g, \gamma < 0$

C. WEAPON EFFECTIVENESS:

1. P(F) and P(S)

2. $E(H) = P(F) + (N - 1)\, P(S)$

SLIDE #10

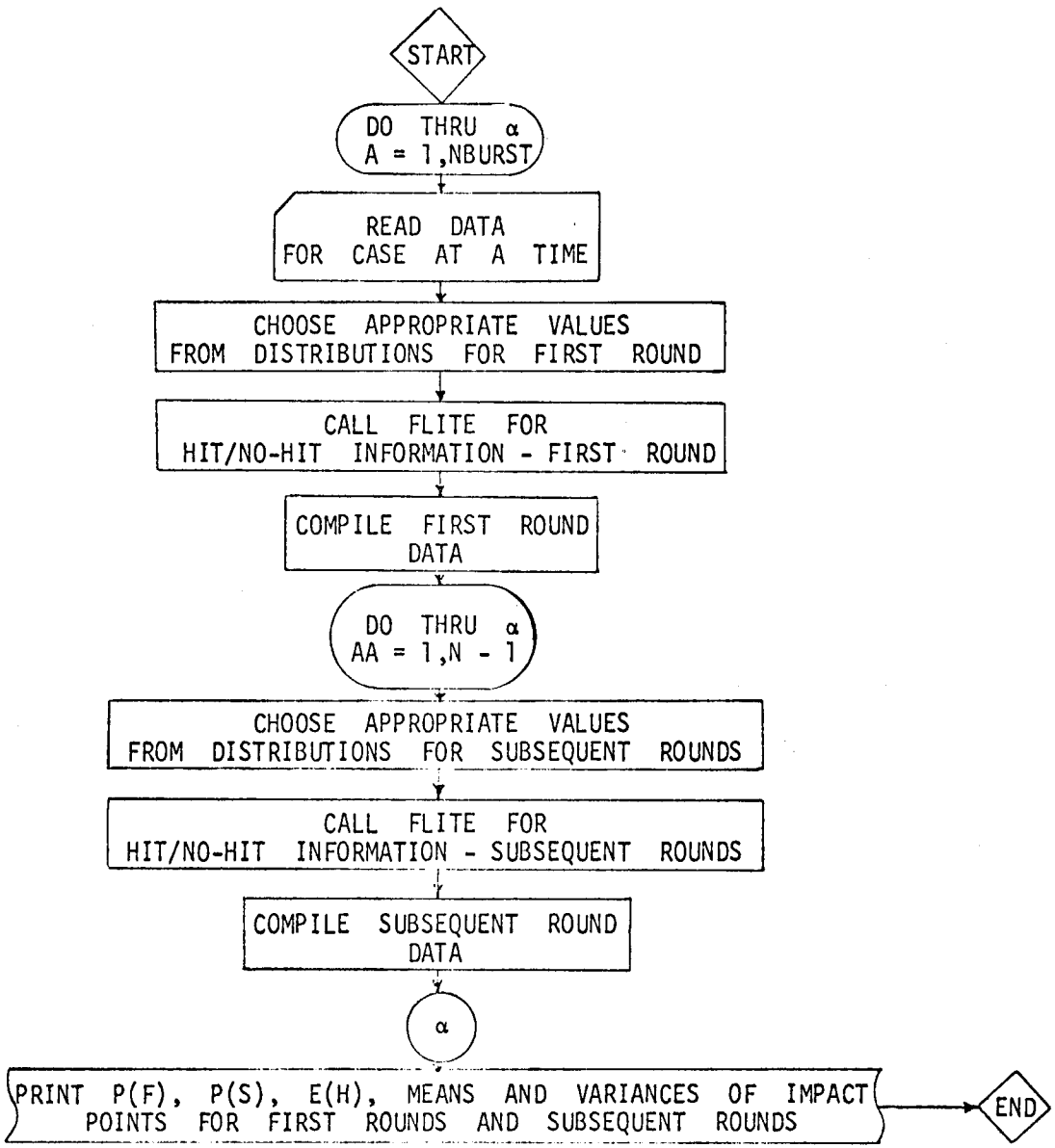
(3) Computer Program Discussion

(a) On the next slide we begin to cover some of the program discussion.

(b) The entire program is written in FORTRAN IV and consists of one MAIN routine, two SUBROUTINES, and one FCTN routine. There is a major SUBROUTINE called FLITE which basically furnishes hit/no-hit data when inputted with some flight data. The other subroutine and FCTN routine just generate normal random numbers.

(c) This entire program is set up to perform and analyze a given number of bursts. It reads the data cards for a case at a time for the five sets of normal-distribution parameters, and other parameters as well. Then it simulates the initial conditions of first round of a burst and subsequently analyzes the flight equations. The FLITE subroutine iterates the time parameter using the Newton-Raphson method in order to converge on a hit/no-hit decision. The program compiles some first-round data and then simulates the subsequent rounds in the same manner. At the conclusion of the simulated bursts (e.g., 1000), then it computes and prints P(F), P(S), E(H), means and variances of impact points for first rounds and subsequent rounds.

III. COMPUTER PROGRAM DISCUSSION



(4) Numerical Examples

(a) As an illustration of the model, [See the next slide #12] consider a MG firing on a hemisphere of six feet radius with burst-length of six rounds for a total of 200 bursts (i.e., 200 first rounds, and 1000 subsequent rounds).

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IV. NUMERICAL EXAMPLE

A. SITUATION

TARGET: HEMISPHERE WITH 6 FEET RADIUS

BURST LENGTH: 6 ROUNDS

NUMBER OF BURSTS: 200

B. ENVIRONMENTAL CONDITIONS

C. FIVE (5) CASES AND THEIR RESULTS

SLIDE #12

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(b) Suffice it to say that some representative set of environmental conditions were used.

gravit. const. accel. (G) = 32.175 ft/sec^2

atmos. density (P) = 2.377×10^{-3} $slug/ft^3$

caliber of proj (D) = 1.9685×10^{-2} ft

proj mass (XM) = 3.774×10^{-4} slug

vel of sound (A0) = 1120.27 ft/sec

(c) I have five example cases to illustrate use of the model. [See the last slide #13]. The distance to target in the first four cases is about 800 meters (2400 ft), whereas in the fifth case, the target was about 1600 meters (4800 ft).

SLIDE #13

SOME RESULTS FOR EXAMPLE CASES

	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5
P(F)	0.8650	0.9550	0.9650	1.0000	0.4700
P(S)	0.8280	0.8700	0.8990	1.0000	0.4540
E(H)	5.0050	5.3050	5.4600	6.0000	2.7400
TAVG	0.913	0.916	0.917	0.918	1.823

(d) The first case was the standard case.

(e) The second and third cases experienced a progressively lesser velocity dispersion in the downrange direction for all rounds of a burst. This had the effect of increasing the P(F), P(S) and E(H) from the "tight" velocity dispersion in the second case to the tighter velocity dispersion in the third case.

(f) In case four, the position and velocity dispersion for all rounds and the offset distance was tightened by two orders of magnitude. Obviously this had to have tightened the shot group per burst, and it reflects so in the results - case four.

(g) In case five, the standard case was used except the distance to target was doubled (from 2400 ft to 4800 ft). Note the average flight time is doubled. As you can see, the projectiles at least made it to the target and had some limited degree of success.

(5) Conclusions

(a) This study is specifically designed for analyzing the effectiveness of small-caliber MG's. In order to easily study all types of MG's with this model, a few minimum changes would have to be incorporated, and a provision for fire control would have to be made.

(b) However, this MG effectiveness study should have illustrated that a complex weapon-mount problem can be addressed in three-dimensions, and the projectiles from that weapon system can be related downrange to a three-dimensional target.

AN ANALYTICAL APPROACH FOR SOME AIR SCATTERABLE MINEFIELD EFFECTIVENESS MODELS

Barry H. Rodin
Ballistic Research Laboratories
Aberdeen Research and Development Center
Aberdeen Proving Ground, Maryland

I. INTRODUCTION. The development of within minefield models for assessing the effectiveness of antivehicular and antipersonnel air scatterable minefields is of current interest to the U. S. Army Materiel Systems Analysis Agency (AMSAA). One such set of models, using a Monte Carlo approach, is presented in Reference [1].

The purpose of this current report is to present analytical closed-form models for this problem. The advantages of the closed-form methodology as compared to a Monte Carlo approach are the reduced computer running time and increased accuracy.

ASSUMPTIONS. In developing the computational procedures we shall make use of the following assumptions:

* Mines are randomly distributed about their droppoints according to any given probability distribution (typically uniform or bivariate normal).

* The actual droppoint for each cluster of mines is distributed about its theoretical aimpoint according to any given probability distribution (typically bivariate normal). This is used to account for error in delivery of the mines.

* Targets breach the minefield in straight-line paths in one or more columns.

* The breach attempt is continued without regard to the number of casualties taken; the targets can not change tactics.

* There are no sympathetic detonations of mines.

* A target can not be incapacitated by a mine which is detonated by another target.

* All mines are of the same type.

* When a target encounters a mine it is either incapacitated so that it can not proceed, or it continues on its path as if the encounter had not occurred; i.e., there is no partial incapacitation.

* Different targets in the column may have different probabilities of detecting a mine and may have different probabilities of being incapacitated by a mine.

* Terrain is not considered.

Input

Input data required by the procedure include the number of droppoints, the number of mines at each droppoint, locations of the theoretical aimpoints, probability distributions of the actual droppoint about each aimpoint, and probability distributions of the mines about each droppoint. In addition, the probability that a mine is a dud must be specified, and, for each target, the probability of detecting a mine previous to encountering it, and the probability that the target is incapacitated when it detonates a mine. The width of the path covered by the target must be specified, and, for the personnel case, the ratio of the area covered by the footsteps to the area of the path.

Output

The computations give, for either a specified breach point or the average over several possible breachpoints, the probability that the $i^{\underline{th}}$ target in the column is incapacitated, as well as the expected number of mines encountered, detected and removed harmlessly, found to be duds, and exploded without incapacitating the target.

Outline of the Computational Procedure

The computation is performed in two parts. The first part finds the probabilities of having exactly $j = 0, 1, 2, \ldots, N$ live mines in the path of the targets before any breach is attempted. The computations allow for the possibility that the actual droppoints differ from the theoretical aimpoints due to aiming error, and the combined effect of mines from all of the droppoints is computed.

The second part of the computation uses the data obtained in the first part to compute the probability that the first target successfully breaches the minefield, as well as the probabilities that exactly $\nu = 0, 1, 2, \ldots, N$ mines remain in the path after the breach attempt. Then, these new probabilities are used to compute the values for the second target in the column, and so on for the remaining targets in the column.

II. SCATTERING THE MINES

Reduction to One Dimension

Although this problem concerns a two-dimensional minefield the computations need only involve one dimension. This is possible since the targets cross the field in a straight path.

For the vehicular case, since the tracks go straight through the minefield, in order to determine if a mine is in the vehicle's path we need only consider the x coordinate (see Figure 1). The y coordinate of the location of a mine is irrelevant in determining if it is in the path of the vehicle.

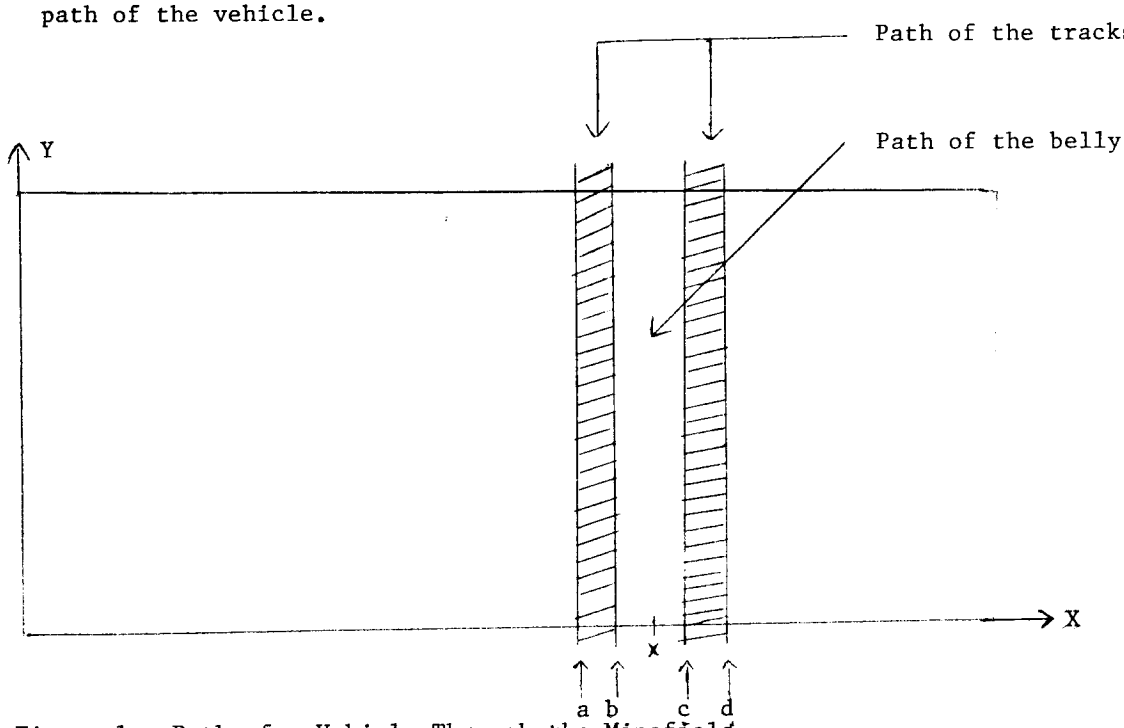

Figure 1. Path of a Vehicle Through the Minefield

For the personnel case we assume that the area covered by a cluster of mines is large in comparison with a man's stride. The average number of mines he would encounter by stepping completely through the minefield is thus approximated by the number of mines in his path times the ratio of the area of his footsteps to the area of the path (see Figure 2).

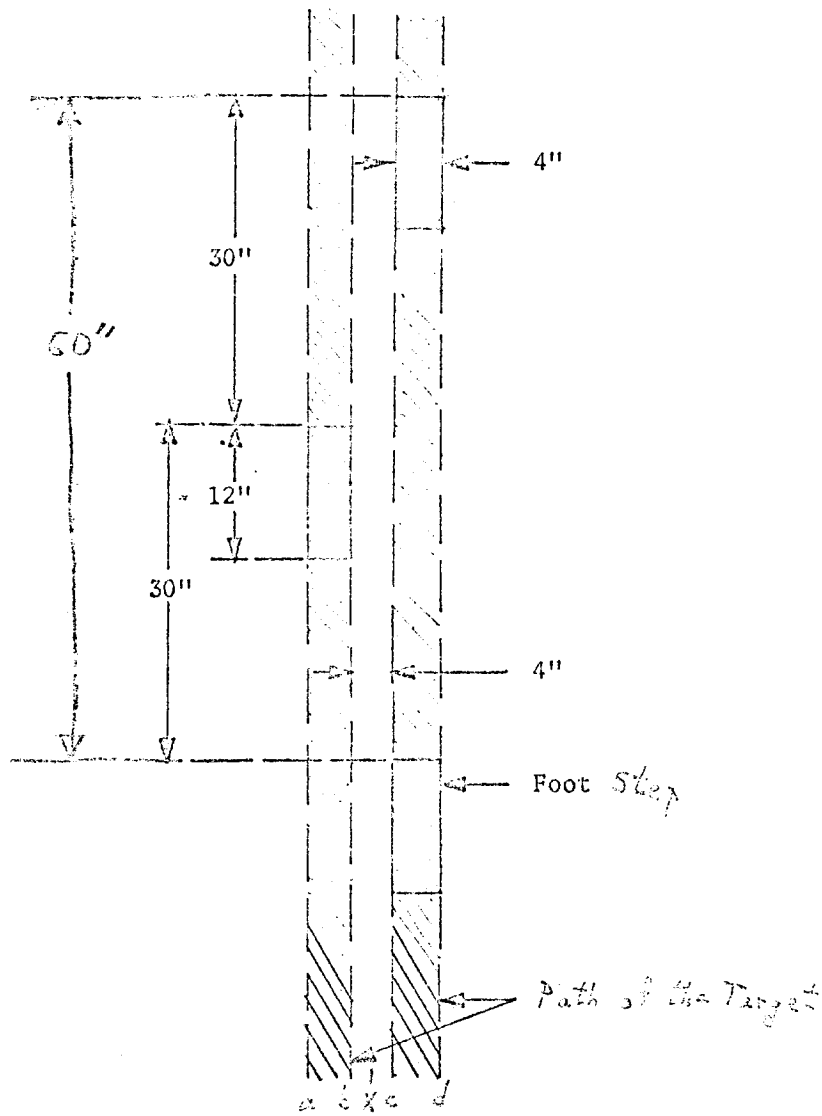

Figure 2. Footstep Pattern

One Aimpoint and No Delivery Error

Let f(x,y) be the probability density that any given mine in the cluster lands at the point (x,y). Since the targets go straight across the minefield parallel to the y axis we need only consider the marginal distribution.

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy.$$

The probability F(x) of a mine landing in the path of the target centered at x is given by expressions of the form:

Vehicular Case:

$$F(x) = \int_a^b f(x)dx + \int_c^d f(x)dx \qquad \text{(Tracks only)}$$

$$F(x) = \int_a^d f(x)dx \qquad \text{(Tracks and Belly)}$$

$$F(x) = \int_b^c f(x)dx \qquad \text{(Belly only)}$$

where a,b.c and d are as ahown in Figure 1.

Personnel Case:

$$F(x) = A \left(\int_a^b f(x)dx + \int_c^d f(x)dx \right)$$

where a,b, c and d are as shown in Figure 2 and where A is the ratio of the area of the footsteps to the area of the path. For example, in Figure 2, b-a = d-c = 4 inches and A = 12/60 = .2.

If there are N mines in the cluster, the probability of exactly j mines in the path at x is

$$C_j(x) = \binom{N}{j} (F(x))^j (1 - F(x))^{N-j}, \quad j = 0, 1, 2, \ldots, N. \qquad (1)$$

To speed up the computations on a computer and to avoid overflow in evaluating the number of combinations we may use the relations

$$C_o(x) = (1 - F(x))^N$$

$$C_j(x) = C_{j-1}(x) \frac{N-j+1}{j} \frac{F(x)}{1-F(x)}, \quad j = 1, 2, \ldots, N.$$

which are easily derived from (1). As a check on the computations note that we must have

$$\sum_{j=0}^{N} C_j(x) = 1.$$

Throughout this paper it will not actually be necessary to carry out the summations up to the number of mines N. In the numerical implementation of these procedures the summations may be terminated as soon as the sum becomes close to 1 and the summands become sufficiently small.

Simplifications for Specific Distributions

Instead of evaluating the integral

$$F(x) = \int_a^d f(x)\, dx \tag{2}$$

by general numerical integration procedures we may use simpler methods for the normal and uniform distributions.

If the marginal distribution of the mines along the x axis is normal with mean μ and standard deviation σ we may use a rational approximation to the normal distribution described in [2] p. 932, 26.2.19 and available as a Fortran function FND. Equation (2) thus becomes

$$F(x) = \text{FND}\left(\frac{d-\mu}{\sigma}\right) - \text{FND}\left(\frac{a-\mu}{\sigma}\right).$$

For mines distributed uniformly between $x = \ell$ and $x = r$ we may use the expression

$$F(x) = \max\left\{\frac{\min\{d, r\} - \max\{a, \ell\}}{r-\ell}, 0\right\}.$$

Suppose that the mines are distributed in the following way: The x coordinate is chosen according to a normal distribution with standard deviation σ_1, and the y coordinate is chosen according to a normal distribution with standard deviation σ_2. For simplicity of the exposition we shall assume that the means of the distributions in both x and y are 0. Finally, the point (x,y) is rotated about the origin by θ radians in the counterclockwise direction to determine the point (x', y') where the mine lands. The values x', y' are still expressed in the original coordinate system.

We now wish to determine the marginal distribution of (x', y') along the x axis, which is simply the distribution of x'. The equations describing a counterclockwise rotation about the origin are

$$x' = x \cos \theta - y \sin \theta \tag{3}$$

$$x' = x \sin \theta + y \cos \theta \tag{4}$$

Since cos θ and sin θ are constants, equation (3) shows that the random variable X' is simply a linear combination of the independently distributed normal random variables X and Y. Hence, (see [3] p. 9 or [4] p, 46), X' is normally distributed with mean 0 and standard deviation

$$\sqrt{(\sigma_1 \cos \theta)^2 + (\sigma_2 \sin \theta)^2}.$$

Thus a rotation performed on a bivariate normal distribution simply yields another normal distribution.

Treatment of Delivery Error

Let E (s) be the marginal probability distribution of the delivery error along the X axis. Typically we shall assume E (s) to be normally distributed with some standard deviation σ so that

$$E(s) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\left(\frac{s}{\sigma}\right)^2\right]$$

where s = 0 represents zero delivery error.

The probability of having exactly j mines in the path centered at x, taking into account the delivery error, is given by

$$D_j(x) = \int_{-\infty}^{\infty} E(s) \; P_j(x-s) \, ds$$

where the functions P_j are as defined in (1). When evaluating the above integral numerically it is not necessary to consider those regions where E (s) is zero or close to it. For E (s) normal we would only integrate from, say, s = -5σ to s = +5σ. As a check on the computations note that we must have

$$\sum_{j=0}^{N} D_j(x) = 1.$$

Treatment of Several Aimpoints

Let

n = the number of aimpoints

N_i = the number of mines to be scattered about aimpoint i

N = the total number of mines

$= N_1 + N_2 + \ldots + N_n$

$D_{j,i}$ = the probability that exactly j mines from the i^{th} aimpoint land in the path centered at x.

P_j = the probability that a total of exactly j mines out of all the aimpoints land in the path centered at x.

Then

$$P_j = \sum_{\{j_i\}} \prod_{i=1}^{n} D_{j_i,i}$$

where the sum is taken over all $\{j_i\}$ such that $j_1 + j_2 + \ldots + j_n = j$.

This computation can be simplified as follows:

Let $T_{j,i}$ = the probability that from the first i clusters there are exactly j mines in the path centered at x.

Then $T_{j,1} = D_{j,1}$

$$T_{j,i} = \sum_{\xi=0}^{j} T_{\xi,i-1} \; D_{j-\xi,\, i} \quad , \quad i = 2,3, \ldots,n$$

$$P_j = T_{j,n}.$$

Note that we need not carry out this summation for any aimpoint i with very small or zero probability of having one or more mines in the path centered at x.

Average Effect Over Several Possible Breachpaths

We shall now consider the case where the target is allowed to choose his path at random from among several possibilities.

Let

$P_j(x_i)$ = the probability that exactly j mines out of all the aimpoints land in the path centered at x_i.

P_j = the probability that exactly j mines out of all the aimpoints land in the path of the target, where the path of the target is chosen at random so that it is equally likely to be centered at any of the points, $x_1, x_2, \ldots, x_n$.

Then P_j is simply the average of the $P_j(x_i)$, namely

$$P_j = \frac{1}{n} \sum_{i=1}^{n} P_j(x_i).$$

III. BREACHING THE MINEFIELD

The previous section showed how to compute the probabilities of having exactly j = 0, 1, . . ., N mines in the path of the target before any breach attempt. This section will show the effect upon the minefield caused by the breach attempt. That is, we shall compute the probabilities of having exactly ν = 0, 1, . . ., N mines in the path of the target after its breach attempt is finished. In addition, we compute the probability that this target successfully breaches the minefield, as well as the expected number of encounters, detections, duds encountered, and non-lethal detonations.

Computation for subsequent targets is performed in a similar manner using the probabilities for the number of mines in the path after the previous target has finished its breach attempt.

We define:

PD = the probability that a mine in the target's path will be detected and removed harmlessly.

PDUD = the probability that a mine is a dud incapable of exploding.

PK = the probability that when a mine explodes it will incapacitate the target.

The values of PD and PK may vary from target to target; the value of PDUD remains the same for all targets.

The computational methods will be presented in two parts - first for the vehicular case and then for the personnel case.

VEHICULAR CASE

In this section we shall use the following nomenclature:

P_j = the probability that there are exactly j mines in the target's path before it attempts to breach the minefield.

P'_ν = the probability that exactly ν mines remain after this breach attempt.

r = the probability that a mine in the target's path will not incapacitate it.

= the probability that the mine will be detected and removed harmlessly, or is a not detected but is a dud, or will explode but not incapacitate the target.

= PD + (1-PD) [PDUD + (1-PDUD) (1-PK)]

1-r = the probability that a mine in the target's path will incapacitate it.

= (1-PD) (1-PDUD) PK

ψ = the probability that this target successfully breaches the minefield.

The equations required for the computation are as follows:

$$P'_\nu = (1-r) \sum_{j=\nu+1}^{N} P_j r^{j-\nu-1} \quad \text{for } \nu > o$$

since if there are j mines to begin with, and ν mines after this target passes, then the first $j-\nu-1$ mines in the path must not incapacitate the target while the next mine does incapacitate it.

$$P'_o = P_o + \sum_{j=1}^{N} P_j r^{j-1} \tag{6}$$

since if there were previously no mines in the path there will still be no mines in the path, and if there were j mines to begin with and no mines afterward then the first j-1 mines must not incapacitate the target and the last mine may either incapacitate it or not, which occurs with probability 1.

The probability that this target gets through the minefield is

$$\psi = P_o + \sum_{j=1}^{N} P_j r^j. \tag{7}$$

Note that this expression has r^j rather than r^{j-1} since the final mine must also be removed harmlessly in order for this target to successfully breach the minefield.

As a check on the computations, note that we must have

$$\sum_{\nu=o}^{N} P'_\nu = 1$$

In order to lessen the amount of computation required we note that

$$P'_\nu = (1-r) \sum_{j=\nu+1}^{N} P_j \, r^{j-\nu-1}$$

and

$$P'_{\nu-1} = (1-r) \sum_{j=\nu}^{N} P_j \, r^{j-\nu}$$

$$= (1-r) \, P_\nu + r \, (1-r) \sum_{j=\nu+1}^{N} P_j r^{j-\nu-1}$$

Hence, in place of equation (5) we may use the relations

$$P'_{\nu-1} = (1-r) \, P_\nu + rP'_\nu \quad \text{for } \nu = N-1, N-2, \ldots, 3, 2. \tag{8}$$

We shall begin the computation with the equation

$$P'_{N-1} = (1-r) \, P_N \tag{9}$$

and then work backward using equation (8) for $\nu = N-1, N-2, \ldots, 3, 2$ in that order. Finally, we note that

$$P'_o = P_o + \sum_{j=1}^{N} P_j \, r^{j-1}$$

$$= P_o + P_1 + r \sum_{j=2}^{N} P_j \, r^{j-2}.$$

Hence,

$$P'_o = P_o + P_1 + \frac{r}{1-r} \, P'_1. \tag{10}$$

Similarly, we see that

$$\psi = P_o + \sum_{j=1}^{N} P_j r^j$$

$$= P_o + rP_1 + r^2 \sum_{j=2}^{N} P_j r^{j-2}$$

Hence,

$$\psi = P_o + r\left(P_1 + \frac{r}{1-r} P_1'\right). \tag{11}$$

For computer implementation use equations (9), (8), (10), and (11) rather than equations (5), (6), and (7) in order to save the computer time which would be required to perform summations for each value of ν.

We shall now show how to compute the expected number of encounters, detections, non-detected mines which are duds, incapacitations, and non-incapacitating detonations.

The number of incapacitations resulting from the breach attempt of any single target is either 0 or 1; i.e., either the target is incapacitated or it is not. As defined in equation (7), the probability of 0 incapacitations is ψ. Thus, the expected number of incapacitations, EK, is given by

$$EK = 1-\psi.$$

We now wish to compute

ED = the expected number of detections

EDUD = the expected number of mines which are not detected but are encountered by the target and are duds.

ENK = the expected number of non-incapacitating detonations.

In order to obtain the above quantities we shall first compute

ER = the expected number of mines removed without incapacitating the target.

= ED + EDUD + ENK

Exactly j mines are removed harmlessly (i.e., without causing an incapacitation) if there are more than j mines in the path at the beginning of the breach attempt, the first j of them are removed harmlessly, and the next one causes an incapacitation, or by starting with exactly j mines and removing all of them. This occurs with probability

$$P_j r^j + \sum_{\zeta=j+1}^{N} P_\zeta r^j (1-r).$$

Thus, the expected number of mines removed harmlessly ER is given by

$$ER = \sum_{j=o}^{N} j\, r^j \left[P_j + (1-r) \sum_{\zeta=j+1}^{N} P\zeta \right] \qquad (12)$$

The above equation requires a double summation, and hence is computationally inefficient. We shall use the following proposition to simplify this computation.

Proposition: Let $x = r + 2r^2 + 3r^3 + \ldots + nr^n$ where $r = 1$. Then,

$$x = \frac{r[1-(n+1)\, r^n + nr^{n+1}]}{(1-r)^2}$$

Proof: $(1-r)x = r\,(1+r+r^2 + \ldots + r^{n-1} - nr^n)$

Using the well known identity

$$1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$$

and simplifying, we obtain the required result.

By reversing the order of summation in equation (12) we obtain

$$ER = \sum_{j=o}^{N} P_j \left[\sum_{\zeta=o}^{j-1} \zeta r^\zeta \quad (1-r) \quad + jr^j \right] .$$

Using the Proposition we can simplify the expression in brackets as follows:

$$\sum_{\zeta=o}^{j-1} \zeta r^\zeta \quad (1-r) \quad + jr^j$$

$$= (1-r)\, \frac{r[1-jr^{j-1} + (j-1)\, r^j]}{(1-r)^2} + jr^j$$

$$= r\, \frac{(1-r^j)}{1-r}$$

Hence, we obtain the simple relationship

$$ER = \frac{r}{1-r} \sum_{j=1}^{N} (1-r^j)\, P_j . \qquad (13)$$

Since the occurence of detections, duds, and non-incapacitating detonations are independent, we have:

$$ED = ER \frac{PD}{r}$$

$$EDUD = ER \frac{(1-PD)\ PDUD}{r}$$

$$ENK = ER \frac{(1-PD)\ \ (1-PDUD)\ \ (1-PK)}{r}$$

The expected number of encounters, EE, is given by

$$EE = EK + ED + EDUD + ENK.$$

PERSONNEL CASE

We assume that the targets follow a straight line path across the minefield and that the footsteps of the current target are random with respect to those of the previous targets. For the personnel case we use the following nomenclature:

P_j = the probability that there are exactly j mines in the target's path before it attempts to breach the minefield

P'_ν = the probability that exactly ν mines remain after this breach attempt

r = the probability that a mine is removed from the path without incapacitating the target

b. = the probability that a mine incapacitates the target

1-r-b = the probability that a mine is stepped over by the target and hence remains in the path of subsequent targets

ψ = the probability that this target successfully breaches the minefield.

The footsteps of all the targets in the same column are constrained to lie within a straight-line path across the minefield. Foe example (see Figure 2), we might consider a footstep to be 4 inches wide and 12 inches long with an average stride length of 30 inches. The path is then the two 4 inch strips corresponding to all the positions in which the target might possibly step. For this example, the ratio of the area of the footsteps to the area of the path is

$$RATIO = \frac{12 \text{ inches}}{60 \text{ inches}} = .2$$

In general we define:

RATIO = the ratio of the area of the target's footsteps to the area of his path

For the personnel case we may proceed under two different assumptions. We may assume either that the target will detect and remove mines only if they are in his footsteps or that the target will detect and remove mines anywhere in the path.

If the target detects and removes mines anywhere in his path, then

$$r = PD + (1-PD)\ (RATIO)\ [PDUD + (1-PDUD)\ (1-PK)\ ]$$

$$b = (1-PD)\ (RATIO)\ (1-PDUD)\ PK$$

$$1-r-b = (1-PD)\ (1-RATIO)$$

If the target detects and removes mines only within his footsteps, then

$$r = (RATIO)\ \{PD + (1-PD)\ [PDUD + (1-PDUD)\ (1-PK)\ ]\ \}$$

$$b = (RATIO)\ (1-PD)\ (1-PDUD)\ PK$$

$$1-r-b = 1-RATIO$$

The values of PD, PK and RATIO may vary from target to target in the same column. The value of PDUD remains the same for all targets.

The probability P'_ν of having exactly ν mines in the path after the breach attempt is given by the equation

$$P'_\nu = P_{\nu_N}\ (1-r-b)^\nu + \sum_{j=\nu H}^{N} P_j \left[\binom{j}{j-\nu} r^{j-\nu}\ (1-r-b)^\nu + b r^{j-\nu-1} \sum_{n=j-\nu}^{j} \binom{n-1}{j-\nu-1}\ (1-r-b)^{n-j+\nu} \right]$$

If 0 mines are removed, all of the mines in the path must be stepped over. If j mines were present before the breach attempt and ν mines afterwards, then $j-\nu$ of the mines must be removed by the breach attempt. If the target is not incapacitated, then $j-\nu$ of the mines are removed harmlessly while the remaining νmines are stepped over and remain on the minefield. On the other hand, if the target is incapacitated by the breach attempt the incapacitation may occur anytime after the target has successfully passed ν mines. Thus, the target may encounter or step over anywhere from $j-\nu$ up to j mines. If the target encounters or steps over exactly η mines then the last of those mines incapacitates the target, $j-\nu-1$ of them are removed harmlessly, and the remining $\eta-j+\nu$ are stepped over.

The probability that this target successfully breaches the minefield is

$$\psi = \sum_{j=o}^{N} P_j\ (1-b)^j \qquad (15)$$

i.e., the target successfully breaches the minefield if he is not incapacitated by any of the mines in his path.

The computation of equation (14) as it stands would be rather time consuming on a computer because of the double summation involved. We now show how to reduce the required computation by computing the inner sum recursively. We define for $\nu = 0, 1, \ldots, N$ and $j = \nu + 1, \nu + 2, \ldots, N$

$$z(\nu,j) = \sum_{\eta=j-\nu}^{j} \binom{\eta-1}{j-\nu-1} (1-r-b)^{\eta-j+\nu}.$$

Hence, by a change of variables,

$$z(\nu-1, j-1) = \sum_{\eta=j-\nu}^{j-1} \binom{\eta-1}{j-\nu-1} (1-r-b)^{\eta-j+\nu}$$

for $\nu = 1, 2, \ldots, N+1$ and $j = \nu+1, \nu+2, \ldots, N+1$. Thus,

$$z(\nu,j) = z(\nu-1, j-1) + \binom{j-1}{j-\nu-1} (1-r-b)^{\nu}$$

for $\nu = 1, 2, \ldots, N$ and $j = \nu+1, \nu+2, \ldots, N$. To begin the computation, note that $Z(o,j) = 1$ for $j=1, \ldots, N$.

Hence,

$$P'_{\nu} = P_{\nu} (1-r-b)^{\nu} + \sum_{j=\nu+1}^{N} P_j \, r^{j-\nu-1} \left[r \binom{j}{j-\nu} (1-r-b)^{\nu} + b \, z(\nu,j) \right] \tag{16}$$

for $\nu = 0, 1, \ldots, N$. Thus, we have reduced a formula involving a double summation to one involving just a single summation. Additional methods for improving the efficiency of the computation are included in the computer program.

We shall now show how to compute the expected number of encounters, detections, non-detected mines which are duds, incapacitations, and non-incapacitating detonations.

The number of incapacitations resulting from the breach attempt of any single target is either 0 or 1. As defined in equation (15), the probability of 0 incapacitations is ψ. Thus, the expected number of incapacitations, EK, is given by

$$EK = 1-\psi$$

We now wish to compute

ED = the expected number of detections

EDUD = the expected number of mines which are not detected but are encountered by the target and are duds

ENK = the expected number of non-incapacitating detonations.

In order to obtain the above quantities we shall first compute

ER = the expected number of mines removed without incapacitating the target

$$= ED + EDUD + ENK$$

Using equation (16) we see that the expected number of mines, ER, which are removed without incapacitating the target is

$$ER = \sum_{\nu=1}^{N} \sum_{j=\nu+1}^{N} \left[(j-\nu) \; P_j \; r^{j-\nu} \binom{j}{j-\nu} (1-r-b)^{\nu} + (j-\nu-1) \; P_j \; r^{j-\nu-1} b \; z \; (\nu,j) \right] \tag{17}$$

Since the occurence of detections, duds, and non-incapacitating detonations are independent, we have: If detections may occur anywhere in the path

$$ED = ER \; \frac{PD}{r}$$

$$EDUD = ER \; \frac{(1-PD) \; (RATIO) \; (PDUD)}{r}$$

$$ENK = ER \; \frac{(1-PD) \; (RATIO) \; (1-PDUD) \; (1-PK)}{r}$$

If the target only detects mines which lie within his footsteps:

$$ED = ER \; \frac{(RATIO) \; PD}{r}$$

$$EDUD = ER \; \frac{(RATIO) \; (1-PD) \; PDUD}{r}$$

$$ENK = ER \; \frac{(RATIO) \; (1-PD) \; (1-PDUD) \; (1-PK)}{r}$$

The expected number of encounters, EE, is given by

$$EE = EK + ED + EDUD + ENK.$$

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A MATHEMATICAL THEORY OF MEASURES OF EFFECTIVENESS

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The existence of more than one choice or course of action to satisfy a given set of requirements necessitates a decision. In making a decision one first attempts to learn as much as possible about the implications of each alternative. One then analyzes this information in order to compare choices in an effective fashion and select an optimal or "best" choice relative to the requirements.

In some cases the implications or effects of various choices are relatively easy to determine. In others, information is incomplete, so that the alternatives can be examined only partially. As a result, decisions made under such circumstances involve an element of uncertainty. Clearly the selection of an alternative which is in fact optimal relative to the requirements is directly related to the quality and availability of input information.

When knowledge of the implications of one or more of the choices is incomplete or nonexistent, the usual procedure for obtaining information is to perform a series of experiments. In this case the input information is a function of experimental design and the validity of the associated raw data.

The raw data obtained from an experiment is generally of little value by itself. It must be processed, i.e., converted to information. Having obtained information from raw data, one then tries to formulate a

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mathematical theory or algorithm by which one can rank-order the alternatives. Normally, the rank-ordering is quantitiative in nature; but it need not be. The fact of the rank-ordering is the essential ingredient.

An algorithm for rank-ordering is usually called, in military parlance, a measure of effectiveness. Although the term "measure of effectiveness" seems to be a common one in military operations research papers, it appears that it has never been precisely defined from a mathematical point of view. Hence, the purposes of this paper are: 1) To abstract the fundamental properties of some commonly used measures of effectiveness and develop a viable definition, 2) to investigate some of the consequences of the definition, and 3) to suggest some new types of measures of effectiveness.

Sufficient mathematical machinery now exists to define the concept of measure of effectiveness and explore it in some detail. The techniques to be used come from statistics and certain disciplines of modern algebra; particularly, group theory and linear algebra. The use of the abstract (axiomatic) approach is not merely a pedegogical ploy. A theory will be developed which at once subsumes the common types of measures of effectiveness. Hence, the abstract formulation is appealing from an economy of space point of view.

In developing measures of effectiveness (MOE), it is important that each contribute new information to the solution of the problem. If measure B is a consequence of measure A, it is clear that the employment of measure B will contribute no new information. In a practical application,

however, it may not be immediately clear that one MOE is a consequence of another. Some new techniques will be needed for indicating the relationships between two measures of effectiveness, especially, in the case where a composite measure is derived from several others.

The first step in the development of the theory of MOE comparison is to define and discuss the notion of distance between two measures. The definition is based on the usual Euclidean distance concept, and, hence, gives the subject of measures of effectiveness a geometric flavor. Given the distance function, it is then possible to talk about the variance of two MOE, and to define three types of correlation. The concept of variance is borrowed directly from statistics. The correlations will be called weak, strong, and statistical. Weak and strong correlation will give computational techniques for determining whether one MOE is a consequence of another (in the sense that the rank-ordering induced by the one is consistent with that induced by the other). Statistical correlation is analogous to the usual statistical definition of the term. A geometric interpretation can be given to the various types of correlation.

The abstract formulation of the theory of measures of effectiveness would be a rather tedious and pointless exercise without several classes of examples. Hence, this article will show that measures of effectiveness are quite easy to develop from experiments. Experiment-formulated measures of effectiveness will be seen to exhibit a special property called transitivity. The class of transitive measures is of considerable importance because two transitive measures can be combined linearly to form a new measure of effectiveness. Correspondingly, an important result will be the construction of a strongly correlated transitive measure from an arbitrary measure.

An interesting result from the theory of statistics is that the likelihood ratio test is a measure of effectiveness.[1] Several other examples of measures will be defined and discussed, including the canonical or discrete measure of effectiveness.

A simple example of a measure of effectiveness is appropriate at this point in order to motivate the discussion to follow. In a high-diving contest the ability and form of each participant is judged on a numerical scale (say from 1 to 10). A perfect performance rates a score of 10.0, with lesser performances being rated accordingly in increments of tenths. Having compared each contestant against the standard, the judge is then able to compare them against each other. Thus contestant A is ranked above contestant B if the difference of their respective scores is positive. If it is negative, B is better than A. The result is a rank ordering of the contestants, with the winner being the individual who is ranked above all of his competitors.

The precise numerical value awarded to each performance is of only relative significance; it provides the means for creating the rank ordering. (In fact, very often there is more than one judge. A score of 8.5 from one might be a score of 7.3 from another — the figures are subjective but hopefully the relative values are reasonably consistent.)

The measure of effectiveness employed in the above example is simply the differences of the pairs of numerical scores; it provides a way of ordering the relative abilities of the contestants. It will be seen

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[1] See, for example, Chernoff and Moses, Elementary Decision Theory, Wiley 1959, Page 256 ff.

that the notion of rank-ordering, as discussed above, provides the motivation for the definition of measure of effectiveness. Before the definition is given, some preliminary concepts are necessary.

LOGICAL PRELIMINARIES

1.1 Definition: Let S be an arbitrary non-empty set. An equivalence relation on S is a relation $\sim$ such that for x, y, z ϵ S:

a) $x \sim x$ for all x (reflexive)

b) $x \sim y$ implies $y \sim x$ (symmetric)

c) $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitive)

It is a well known fact of elementary set theory that a relation on S is an equivalence relation if and only if it partitions S into mutually exclusive and exhaustive subsets; that is if and only if there exists a collection $\{S_x: x \in I\}$ induced by $\sim$ such that $\bigcup_{x \in I} S_x = S$ and $S_x \cap S_y = \emptyset$ if $x \neq y$ (7, p. 30). The $\{S_x\}$ are obtained by taking $S_x = \{y: y \sim x\}$. Each $x \in S$ produces such an S_x, and for any x, y ϵ S, either $S_x \cap S_y = \emptyset$ or $S_x = S_y$. The index set I is formed by taking a representative of each distinct S_x.

1.2 Definition: Let S be an arbitrary nonempty set. A relation $\leq$ on S is a partial order if:

a) $x \leq x$ for all $x \in S$ (reflexive)

b) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive)

c) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetric).

In addition, $\leq$ is a linear order if:

d) for each x, y ϵ S, $x \leq y$ or $y \leq x$.

Notice that $x \leq y$ may or may not imply that $x = y$. In the case where $x \leq y$ and it is necessary to indicate that $x \neq y$, the notation $x < y$ will be used; that is $x < y$ means $x \leq y$ and $x \neq y$.

MEASURE OF EFFECTIVENESS

In this section the definition of measure of effectiveness will be presented. A preliminary study of a mathematical structure called a linear O-Group is required. The notion of a linear O-Group is sufficiently general to permit simultaneously the study of the two most common types of measures of effectiveness, to be presented below.

2.1 Definition: Let G be a group (see 6 p. 6). An operator group or O-group is an ordered pair (G,O) where O is a set called the operator set, together with a function *: O x G $\longrightarrow$ G, satisfying the following:

$\forall x, y \in G$ and $s \in O$, $s * (xy) = (s * x)(s * y)$ (See 6 p. 40).

2.2 Definition: Let G be an O-group. Then G is said to be a linear O-group if:

a) A linear order $\leq$ is defined on G, as in 1.2.

b) $\forall x,y,z \in G$, $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$.

Comment on Notation: An interesting mathematical problem is whether or not every linear O-group is Abelian (commutative). However, the examples to be discussed in this paper will all involve Abelian groups, so that a further condition will be imposed: Namely for $\forall x, y \in G$, $xy = yx$. Since it is more intuitively appealing to use the additive notation $(x + y = y + x)$ in the case of Abelian groups, this notation will be employed throughout the remainder of the paper.

Examples: The first example of a linear O-group is $((\underline{R},+), \underline{R})$ the group of additive real numbers with the real numbers $\underline{R}$ being the operator set. (Alternatively the ring of integers or field of rational numbers could

be used as an operator set.) The linear ordering will be the usual ordering of $\underline{R}$. Since for $x, y, z \in \underline{R}$ the identity $x(y + z) = xy + xz$ holds, the condition of definition 2.1 is satisfied. Hence $((\underline{R}, +), \underline{R})$ is an $\underline{R}$ - group, where the operation * is real - number muliplication. The second important example of an O-group is $((\underline{R}^+, \cdot), \underline{R})$; namely, the group of positive multiplicative real numbers. Again, the operator set O will be $\underline{R}$ (or integers or rational numbers). The operation * will be exponentiation. Hence for $x, y \in \underline{R}^+$, $z \in \underline{R}$, $z * (xy) = (xy)^z = x^z y^z = (z * x)(z * y)$. This proves that $((\underline{R}^+, \cdot), \underline{R})$ together with exponentiation is a linear O-group.

For the purpose of the discussion to follow it is necessary to make an observation about linear O-groups.

2.3 Proposition: Let G be a linear O-group. Then G has no non-zero elements of finite order. This is, there is no $x \in G$ with $x \neq 0$ and $nx = 0$ for some positive integer n.

Proof: The first step is to show that there is no nonzero element of order two. Assume that $x + x = 0$. It will be shown that $x = 0$. Without loss of generality, suppose $0 \leq x$. Then $0 + x \leq x + x$, or $x \leq 0$. By definition 1.2, $0 \leq x$ and $x \leq 0$ imply that $x = 0$. To show that there is no n such that $nx = 0$, proceed by mathematical induction. Suppose for all positive $r < n$ that $0 < rx$. Then $0 + x < (n - 1)x + x$ in particular, i.e. $x < nx$. Since $0 < x$ it follows that $0 < nx$ by transitivity.

2.4 Definition: Let G be a linear O-group and S a non-empty set. A measure of effectiveness (MOE) is a function $M: S \times S \longrightarrow G$ such that:

a) $\forall x,y \in S$, $M(x,y) = -M(y,x)$

b) $\forall x,y,z \in S$, $M(x,y) \leq 0$, $M(y,z) \leq 0$ implies $M(x,z) \leq \min \{ M(x,y), M(y,z) \}$.

A measure of effectiveness will be transitive if:

b-1) $\forall\, x,y,z \in S,\ M(x,y) + M(y,z) = M(x,z)$.

A few comments on the definition are useful. The significance of a measure of effectiveness is that it provides a method of rank-ordering the elements of S. Here, of course, the set S is intended to be the set of objects or alternatives for comparison.

The condition $M(x,y) = -M(y,x)$ says that if y is preferred to x then x is not preferred to y, and the magnitude is the same in each case. The condition $M(x,y) \leq 0,\ M(y,z) \leq 0,\ M(x,z) \leq \min \{ M(x,y), M(y,z) \}$ says two things. First, it says that if y is preferred to x and z is preferred to y, then z is preferred to x. Second, it indicates a cardinal relationship between x, y, and z. That is, the value by which z is preferred to x is at least as great as the value by which y is preferred to x and the value by which z is preferred to y. It is clear that condition b-1 implies condition b.

As indicated in the introduction, transitive measures of effectiveness are extremely important, because they are closed under addition (i.e. the sum of two transitive MOE's is a transitive MOE). A method will be developed, given an arbitrary measure of effectiveness, for deriving a strongly correlated transitive MOE.

The construction of the rank-ordering will be discussed in depth presently. For logical consistency, two preliminary results are needed.

2.5 Proposition: $\forall\, x,y,z \in S,\ 0 \leq M(x,y),\ 0 \leq M(y,z)$ implies $\max \{ M(x,y), M(y,z) \} \leq M(x,z)$.

Proof: Assume $0 \leq M(x,y)$ and $0 \leq M(y,z)$. Then $-M(x,y) = M(y,x) \leq 0$ and $-M(y,z) = M(z,y) \leq 0$. Hence $M(z,x) \leq \min \{M(z,y), M(y,x)\} \Rightarrow M(z,x) \leq M(z,y)$, $M(z,x) \leq M(y,x) \Rightarrow -M(z,y) \leq -M(z,x)$, $-M(y,x) \leq -M(z,x) \Rightarrow M(y,z) \leq M(x,z)$, $M(x,y) \leq M(x,z) \Rightarrow \max \{M(x,y), M(y,z)\} \leq M(x,z)$.

2.6 Proposition: Let $x,y,z \in S$ be such that $M(x,y) = 0$ and $M(y,z) = 0$. Then $M(x,z) = 0$.

Proof: Since $M(x,y) = M(y,z) = 0$ it follows by definition 2.4 and proposition 2.5 that:

$$M(x,z) \leq \min \{M(x,y), M(y,z)\} = \max \{M(x,y), M(y,z)\} = 0 \leq M(x,z).$$

Hence $M(x,z) = 0$.

Given a MOE, the technique for extracting the associated ranking on S is straightforward. It is at this point that the definitions of section 1 on linear orderings become useful. An equivalence relation $\equiv$ is induced on S as follows:

2.7 Theorem: Let $M: S \times S \longrightarrow G$ be a MOE. Say that $x \equiv y$ for $x,y \in S$ if $M(x,y) = 0$. Then the relation $\equiv$ is an equivalence relation on S.

Proof: Let $x,y,z \in S$. Since $M(x,y) = -M(y,x)$ for all $x,y \in S$, it follows that $M(x,x) + M(x,x) = 2M(x,x) = 0$, so that by theorem 2.3, $M(x,x) = 0$. This proves the reflexivity of $\equiv$. If $M(x,y) = 0$ then $M(y,x) = -M(x,y) = 0$, so $\equiv$ is symmetric. To show transitivity, assume that $M(x,y) = M(y,z) = 0$. Then by proposition 2.6, $M(x,z) = 0$. It follows that $x \equiv y$ and $y \equiv z$ implies $x \equiv z$.

Let $\{S_x: x \in I\} = \mathcal{S}$ be the partition on S induced by $\equiv$ (see definition 1.1), where I is an index set obtained by taking one representative of each equivalence class. The next step in constructing the ordering on S is to linearly order the set $\mathcal{S}$. For this purpose, a definition is needed.

2.8 Definition: Let S_x, S_y be two elements of $\mathcal{S}$. Then say $S_x \lesssim S_y$ if $M(x,y) \leq 0$. The relation $\lesssim$ is well defined: Suppose $M(x,y) \leq 0$. Pick $x_1 \in S_x$ and $y_1 \in S_y$. Then $M(x_1, x) = 0$ so that $M(x_1, y) \leq 0$. Similarly $M(y, y_1) = 0$ so that $M(x_1, y_1) \leq 0$. Thus the definition of $\lesssim$ does not depend upon the particular representatives of S_x and S_y chosen.

2.9 Proposition: The relation $\lesssim$ is a linear ordering on $\mathcal{S}$.

Proof: a) Clearly $S_x \lesssim S_x$ since $M(x,x) = 0 \Rightarrow M(x,x) \leq 0$.

b) Suppose for $x, y \in S$, $S_x \lesssim S_y$ and $S_y \lesssim S_x$. Then $M(x,y) \leq 0$ and $M(y,x) \leq 0 \Longrightarrow M(x,y) = 0 \Longrightarrow S_x = S_y$.

c) Assume that for some $x, y, z \in S$ the relations $S_x \lesssim S_y$, $S_y \lesssim S_z$ hold. Then $M(x,y) \leq 0$ and $M(y,z) \leq 0 \Longrightarrow M(x,z) \leq 0 \Rightarrow S_x \lesssim S_z$.

d) The final step is to verify that for each $x,y \in S$, $S_x \lesssim S_y$ or $S_y \lesssim S_x$. But by definition, $M(x,y) = -M(y,x)$. Since $M(x,y)$ and $M(y,x)$ are both elements of the linear group G it follows that either $M(x,y) \leq 0$ or $0 \leq M(x,y)$. In the latter case, $M(y,x) \leq 0$. Thus $S_x \lesssim S_y$ or $S_y \lesssim S_x$, depending on whether $M(x,y) \leq 0$ or $M(y,x) \leq 0$.

2.10 Definition: Let $M: S \times S \longrightarrow G$ be a MOE, and $\lesssim$ be the induced linear ordering on the set $\mathcal{S}$ of equivalence classes (as defined in 2.8). Then for $x, y \in S$, one says that y is preferred to x (denoted $x < y$) if $S_x \lesssim S_y$ and $S_x \cap S_y = \emptyset$. Similarly one says that x and y are tied if $S_x = S_y$. The combined symbol $x \leq y$ will denote the fact that x is not preferred to y.

It is now possible to prove two propositions of the same nature as 2.5 and 2.6.

2.11 Proposition: Suppose for $x,y,z \in S$ that $M(x,y) \leq 0$, $M(y,z) \leq 0$, and $M(x,z) = 0$. Then $M(x,y) = M(y,z) = 0$.

Proof: By definition, $0 = M(x,z) \leq \min \{M(x,y), M(y,z)\} \Rightarrow 0 \leq M(x,y)$ and $0 \leq M(y,z)$. But $M(x,y) \leq 0$, and $M(y,z) \leq 0$ so that $M(x,y) = M(y,z) = 0$.

Proposition 2.11 provides an important fact from a logical point of view: the knowledge that y is preferred to x, z is preferred to y, and x is tied with z are sufficient to conclude that x, y, and z are all tied.

2.12 Proposition: Let $x,y,z \in S$ and suppose that $M(x,y) = 0$. Then $M(x,z) = M(y,z)$.

Proof: Without loss of generality, assume that $0 < M(y,z)$. Then $\max \{M(x,y), M(y,z)\} = M(y,z) \leq M(x,z)$. In particular, $0 < M(x,z)$. Since $M(x,y) = -M(y,x) = 0$ it follows that $\max \{M(y,x), M(x,z)\} = M(x,z) \leq M(y,z)$. Hence $M(x,z) = M(y,z)$.

This proposition gives a consistency relation for a measure of effectiveness. If two objects are ranked equally the relation of either of them to any third object is identical.

The scheme just discussed gives a method of inducing a preference ranking on the objects of a nonempty set S, by the use of a measure of effectiveness.

Basically an object y is preferred to another object x if $M(x,y) < 0$ and x is tied with y if $M(x,y) = 0$. Observe that the induced ranking has the following properties:

a) Every object of S may be compared with every other object.

b) If y is preferred to x and z is preferred to y, then z is preferred to x.

c) If y is preferred to x and tied with z, then z is preferred to x.

d) If x and y are tied and y and z are tied, then x and z are tied.

The chief purpose of a measure of effectiveness is to generate the associated rank-ordering. However, the values assumed by the MOE may possess intrinsic significance. That is, the quantification may answer the question: "To what extent is y preferred to x?" In any case the MOE gives a mathematical basis for making decisions: if y is preferred to x, select y; among several candidates choose a maximal one. The decision problem becomes one of careful selection of the measure(s) of effectiveness. Having established the MOE, the decision is a logical consequence.

The values assumed by a measure of effectiveness become important when the validity of the measure is questioned. For example if a MOE is derived from a statistical analysis of data, the values obtained may be correct only to within a certain confidence interval. Hence one may require an "aspiration level" condition to be met before one accepts the validity of a rank-ordering.

The derivation of a technique for factoring the confidence interval out of a measure of effectiveness which contains an uncertainty constitutes an important unsolved problem.

A measure of effectiveness is of no value in differentiating between two alternatives if it declares them to be tied. In such a situation one looks for further evidence — usually another MOE. One hopes that a second MOE will rank the two tied objects. However, a potential problem arises with the introduction of a second MOE. For illustration, suppose that two measures M and N are given. Suppose further that for two objects x, y ϵ S, measure M prefers x to y and measure N prefers y to x. Clearly a contradiction arises; additional information is required in order to choose between x and y. As illustrated in the example, comparison of two MOE may create new difficulties at the same time that it resolves old ones. A technique for resolving contradictions, given additional information, will now be discussed.

MEASURE PRESERVING COMPOSITIONS

The problem discussed above requires, for its solution, the formation of new measures of effectiveness out of old ones. This section will present some theorems describing situations under which the MOE property is preserved or (in mathematical language) invariant.

3.1 Definition: Let G, H be linear O-groups. A function $\emptyset$: G$\longrightarrow$ H will be called measure preserving if:

a) $\emptyset(-x) = -\emptyset(x)$, $\forall x \in G$

b) $\forall x, y \in G$, $x < y \Longrightarrow \emptyset(x) < \emptyset(y)$.

3.2 Proposition: Let $\emptyset$: G H be measure preserving. Then

a) $x < 0 \Rightarrow \emptyset(x) < 0$

b) $0 < x \Longrightarrow 0 < \emptyset(x)$

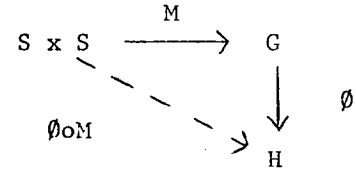
c) $\emptyset(0) = 0$

Proof: Parts a) and b) are immediate consequences of 3.1 b). Part c) follows from $\emptyset(-0) = -\emptyset(0) \Longrightarrow 2\emptyset(0) = 0$. By proposition 2.3, $\emptyset(0) = 0$.

The following theorem is an immediate consequence of 3.1.

3.3 Theorem: Let S be a nonempty set and let G, H be linear O-groups. Let $M: S \times S \longrightarrow G$ be a MOE and $\emptyset: G \longrightarrow H$ a measure preserving function. Then the composition $\emptyset oM: S \times S \longrightarrow H$ defined by $\emptyset oM(x,y) = \emptyset(M(x,y))$ is a MOE.

Proof: The following diagram is a useful mnemonic:



It is necessary to verify the following conditions:

a) $\emptyset oM(x,y) = -\emptyset oM(y,x)$

b) $\emptyset oM(x,y) \leq 0$, $\emptyset oM(y,z) \leq 0 \Longrightarrow$

$\emptyset oM(x,z) \leq \min \{\emptyset oM(x,y), \emptyset oM(y,z)\}$.

Since $M(x,y) = -M(y,x)$, it is clear that $\emptyset oM(x,y) = \emptyset(M(x,y)) = \emptyset(-M(y,x)) = -\emptyset(M(y,x)) = -\emptyset oM(y,x)$.

Now assume that $\emptyset oM(x,y) \leq 0$ and $\emptyset oM(y,z) \leq 0$. In the case where $\emptyset oM(x,y) < 0$, $M(x,y) < 0$ by definition. If $\emptyset oM(x,y) = 0$, one uses proposition 3.2 c) to conclude that $M(x,y) = 0$. In other words, $\emptyset oM(x,y) \leq 0 \Longrightarrow M(x,y) \leq 0$. Similarly, $M(y,z) \leq 0$ so that:

$$M(x,z) \leq \min \{M(x,y), M(y,z)\}$$

By definition 3.1, $\emptyset oM(x,z) \leq \min \{\emptyset oM(x,y), \emptyset oM(y,z)\}$.

A few words of explanation would be appropriate at this point. A measure preserving function is a generalization of the notion of

multiplication by a positive constant. That is, if the linear group G is in fact the real numbers, or a subfield of $\underline{R}$, then given a MOE $M: S \times S \longrightarrow \underline{R}$, and a $\epsilon \underline{R}^+$ the function $M = a M$ is also a MOE, obtained by scaling M by a. The rationale for scaling a measure of effectiveness is to be able to adjust units, if need be, in order to linearly combine two measures of effectiveness. When linearly combining two MOE's it is important that the units of each be identical. The linear combination of two MOE's will be discussed fully in the next section.

3.4 Theorem. Let $\emptyset: G \longrightarrow H$ be measure preserving. Then for any measure of effectiveness M, $\emptyset$oM and M induce the same rank-ordering.

Proof: It is sufficient to prove that the set of equivalence classes induced by $\emptyset$oM is the same as that induced by M, and that the linear ordering of the classes is stable under $\emptyset$. But $\emptyset oM(x,y) = 0$ if and only if $M(x,y) = 0$. Hence for each $x \epsilon S$, $(S_x,M) = \{y: M(x,y) = 0\} = (S_x,\emptyset oM) = \{y: \emptyset oM(x,y) = 0\}$. Moreover, $\emptyset oM(x,y) < 0$ if and only if $M(x,y) < 0$, so that $(S_x, M) < (S_y, M)$ if and only if $(S_x, \emptyset oM) < (S_y, \emptyset oM)$.

3.5 Theorem: Suppose that G is an ordered field (see 7 p. [illegible]) and suppose that $\emptyset: G \longrightarrow G$ is measure preserving. Then there exists a function $f: S \times S \longrightarrow G$ such that:

a) $\forall x,y \epsilon S$, $\emptyset oM(x,y) = f(x,y) \cdot M(x,y)$

b) $f(x,y) = f(y,x)$, $\forall x,y \epsilon S$

c) $f(x,y) \succ 0$, $\forall x,y \epsilon S$.

Proof: Case 1. Suppose $M(x,y) \neq 0$. Then define $f(x,y) = [M(x,y)]^{-1} \emptyset oM(x,y)$. Since $\emptyset oM(x,y) < 0$ if and only if $M(x,y) < 0$ it follows that $f(x,y) \succ 0$. Also, $f(x,y) = [M(x,y)]^{-1} \emptyset oM(x,y) = [-M(y,x)]^{-1} [-\emptyset oM(y,x)] = [M(y,x)]^{-1} \emptyset oM(y,x) = f(y,x)$.

Case 2. $M(x,y) = 0$. Define $f(x,y)$ arbitrarily, subject to the restrictions that $f(x,y) > 0$ and $f(x,y) = f(y,x)$. Clearly $f(x,y)$ satisfies the requirements of the theorem.

A partial converse is also true.

3.6 Theorem: Let $f: S \times S \longrightarrow G$ be a function which satisfies b) and c) of theorem 3.5. In addition, suppose that:

$$\max \{f(x,y), f(y,z)\} \leq f(x,z), \quad \forall x,y,z \in S.$$

Then for any MOE $M: S \times S \longrightarrow G$, the function $N(x,y) = f(x,y) \cdot M(x,y)$ is a MOE.

Proof: First observe that $N(x,y) = f(x,y) \cdot M(x,y) = f(y,x) \cdot [-M(y,x)] = -f(y,x) \cdot M(y,z) = -N(y,x)$. Next, suppose that $N(x,y) \leq 0$ and $N(y,z) \leq 0$. Since $f(x,y)$ and $f(y,z)$ are positive, it follows that $M(x,y) \leq 0$ and $M(y,z) \leq 0$. By definition, $M(x,z) \leq M(x,y)$ and $M(x,z) \leq M(y,z)$. Using an elementary argument, one verifies that $N(x,z) \leq \min \{N(x,y), N(y,z)\}$.

Theorems 3.5 and 2.6 adequately characterize measure preserving functions with values in an ordered field (subfield of the real numbers).

A somewhat weaker notion than measure preserving is measure respecting, as defined below.

3.7 Definition: Let G, H be linear O-groups. Let $M: S \times S \longrightarrow G$ be a MOE. A function $\emptyset: G \longrightarrow H$ is said to respect the measure M if:

a) $\emptyset(-x) = -\emptyset(x), \quad \forall x \in G.$

b) $x < y \Rightarrow \emptyset(x) \leq \emptyset(y), \quad x,y \in G.$

c) $\emptyset oM(x,y) \leq 0, \ \emptyset oM(y,z) \leq 0 \Longrightarrow$

$\emptyset oM(x,z) \leq \min \{\emptyset oM(x,y), \emptyset oM(y,z)\} \ \forall x,y,z \in S.$

The function $\emptyset$ is measure respecting if it respects the measure M, for each $M: S \times S \longrightarrow G$.

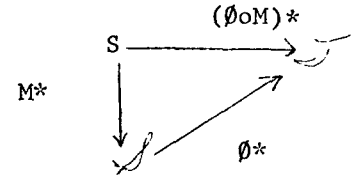
3.8 Proposition: Let $M: S \times S \longrightarrow G$ be a MOE, and $\emptyset: G \longrightarrow H$ a function which respects M. Then the ranking induced by $\emptyset oM$ is consistent with the ranking induced by M.

Proof: The fact that $x < y$ under M implies $x \leq y$ under $\emptyset oM$ for $x, y \in S$, is a trivial consequence of definition 3.7. Moreover, $M(x,y) = 0$ implies $\emptyset oM(x,y) = 0$ by proposition 3.2.

It is immediate from proposition 3.8 that the equivalence classes $(S_x, \emptyset oM)$ induced by $\emptyset oM$ are unions of equivalence classes (S_x, M) induced by M. That is, $(S_x, M) \subset (S_x, \emptyset oM)$ for all x, and $(S_x, M) < (S_y, M) \Rightarrow (S_x, \emptyset oM) \lesssim (S_y, \emptyset oM)$.

Condition 3.5 c) is logically necessary to guarantee that $\emptyset oM$ is a MOE. That is, counterexamples exist to the statement that if $\emptyset: G \longrightarrow H$ satisfies 3.5 a) and b), and M is a MOE, then $\emptyset oM$ is a MOE.

Proposition 3.8 can be summarized pictorially as follows: Let $\mathcal{S} = \{S_x: x \in I\}$ be the set of equivalence classes formed by a measure M. Let $\mathcal{T}$ be $\{T_x: x \in J\}$, where for $x \in S$, $T_x = \{y: \emptyset oM(x,y) = 0\}$, and where J is an index set consisting of one element from each T_x. Then the following diagram is consistent:



where $M^*(x) = S_x$, $(\emptyset oM)^*(x) = T_x$ and $\emptyset^*(S_x) = T_x$. Moreover, $S_x \lesssim S_y \Rightarrow \emptyset^*(S_x) \lesssim \emptyset^*(S_y)$.

The characterization of all functions $\emptyset$ which respect a given measure M is an unsolved problem. A partial answer is provided by the following.

3.9 Proposition: Let M be a measure. Then every measure preserving function respects M.

Proof: Trivial.

LINEAR COMBINATIONS OF MEASURES

Let M_1, M_2 be measure of effectiveness. It would be useful to be able to say that the sum $M_1 + M_2$, defined by $(M_1 + M_2)(x,y) = M_1(x,y) + M_2(x,y)$ is a measure of effectiveness also. Unfortunately this is not always the case. The difficulty occurs in trying to satisfy the transitivity axiom. The following counter example serves to illustrate the point: Let S = {A, B, C}. Suppose two measures M_1 and M_2 are represented in matrix form as[1]

$$M_1 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 5 \\ -1 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix} \end{array} \qquad M_2 = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix} \end{array}$$

Thus, for example, $M_1(A,B) = 1$, $M_1(B,C) = 1$, $M_2(A,C) = -2$. But

$$M_1 + M_2 = \begin{pmatrix} 0 & 1 & 5 \\ -1 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -1 \\ -3 & 1 & 0 \end{pmatrix}$$

Note that $(M_1 + M_2)(A,B) = -1$, $(M_1 + M_2)(B,C) = -1$, but $(M_1 + M_2)(A,C) = 3$. Hence, definition 2.4 b) is not satisfied for $M_1 + M_2$.

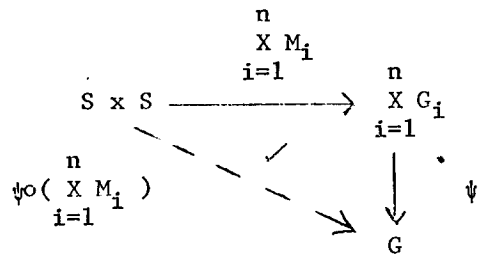
The above example shows that the problem of linearly combining two measures of effectiveness is non-trivial.

---

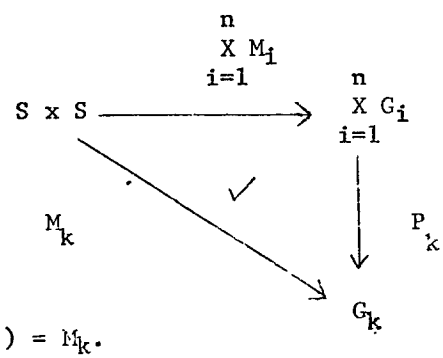
[1]It is useful to observe at this point that a measure of effectiveness may be represented as a square matrix, doubly indexed on the set of objects. This is due to the fact that a measure is a function on pairs. Thus M(x,y) gives the entry in the xth row and yth column. Notice that every MOE can be represented as a skew symmetric matrix, since M(x,y) = -M(y,x). If the matrix is arranged in canonical form with the objects arranged from the top and left in order of descending rank (with a choice of ordering in the case of ties), the skew symmetry becomes immediately apparent.

It is clear from the above example that the combination of MOE's to produce a MOE constitutes a non-trivial problem, even in reasonably restricted cases. As mentioned in the introduction, MOE's which are transitive can be combined linearly. This notion will be developed presently. However, a few preliminary ideas will be presented first.

4.1 Definition: Let $M_i: S \times S \longrightarrow G_i$ be MOE's, where $i = 1,2,\ldots n$ and $\{G_i\}$ are linear O-groups. Let G be a linear O-group and suppose $\exists$ a function $\psi : \underset{i=1}{\overset{n}{X}} G_i \to G$ such that the composition $\psi \circ (\underset{i=1}{\overset{n}{X}} M_i): S \times S \longrightarrow G$, defined by $\psi \circ (\underset{i=1}{\overset{n}{X}} M_i)(x,y) = \psi (\underset{i=1}{\overset{n}{X}} M_i(x,y))$ is a MOE. Then ψ is called a composition function and $\psi \circ (\underset{i=1}{\overset{n}{X}} M_i)$ is a composite MOE. The following triangle illustrates definition 4.1:



Note in particular, if $\psi = P_k$, the kth co-ordinate projection, the following situation results:



since $P_k \circ (\underset{i=1}{\overset{n}{X}} M_i) = M_k$.

Definition 4.1 is sufficiently general that an attempt to characterize composition functions appears to be futile. Some information is available, however, in the special case where the measures $\{M_i\}$ are transitive (see definition 2.4 b-1), and the composed measure is also transitive. In that case one can prove the following.

4.2 Theorem: Let $\{M_i: i = 1;...n\}$ be transitive measures and suppose that $\psi o(\overset{n}{\underset{i=1}{X}} M_i)$ is transitive. Then $\forall x,y,z \in S$,

$$\psi(\overset{n}{\underset{i=1}{X}} (M_i(x,y) + M_i(y,z))) = \psi o(\overset{n}{\underset{i=1}{X}} M_i)(x,y) + \psi o(\overset{n}{\underset{i=1}{X}} M_i)(y,z).$$

Proof: $\psi o(\overset{n}{\underset{i=1}{X}} M_i)(x,y) + \psi o(\overset{n}{\underset{i=1}{X}} M_i)(y,z) = \psi o(\overset{n}{\underset{i=1}{X}} M_i)(x,z) =$ $\psi(\overset{n}{\underset{i=1}{X}} M_i(x,z)) = \psi(\overset{n}{\underset{i=1}{X}} (M_i(x,y) + M_i(y,z)))$, by two applications of the definition of transitivity.

The intent of theorem 4.2 is to show that $\psi: \overset{n}{\underset{i=1}{X}} G_i \longrightarrow G$ behaves like a group homomorphism on the subset $(\overset{n}{\underset{i=1}{X}} M_i)(S \times S) \subset \overset{n}{\underset{i=1}{X}} G_i$, whenever the summation operation makes sense (i.e. on transitive pairs).

4.3 Observation: One might be interested in studying the following situation: Suppose $M_1, M_2, \ldots M_n$ are transitive MOE's. Suppose further that $\psi o (\overset{n}{\underset{i=1}{X}} M_i)$ in a transitive MOE. If the function ψ is in fact a group homomorphism then the MOE $\psi o (\overset{n}{\underset{i=1}{X}} M_i)$ is essentially a linear combination $\overset{n}{\underset{i=1}{\Sigma}} \psi_i (M_i)$ of the component MOE's. That is, if ψ is a group homomorphism, the MOE composition problem can be solved in essentially only one way. This fact will be explored in theorem 4.5. One might wonder, in view of theorem 4.2, if it is sufficient to assume that ψ is a group homomorphism. That is, given a function ψ as defined above, can one always construct a group homomorphism $\Phi : \overset{n}{\underset{i=1}{X}} G_i \rightarrow G$ such that $\psi o (\overset{n}{\underset{i=1}{X}} M_i) (X, Y) = \Phi o (\overset{n}{\underset{i=1}{X}} M_i) (X, Y)$, $\forall$ X, Y $\in$ S. One can show by counterexample that such is not always the case. Thus, the enunciation of sufficient conditions for ψ to be "essentially" a group homomorphism currently constitutes an interesting unsolved problem.

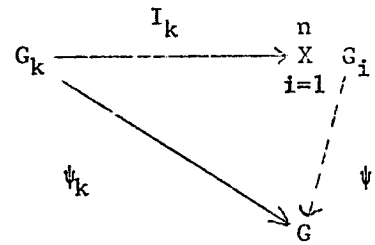
The following theorem gives a converse to observation 4.3, in the sense that it shows a method of developing a composite homomorphism which is a composition function, given the components. A lemma is needed.

4.4 Lemma: Let G, H be linear O-groups, and let $M: S \times S \longrightarrow G$ be a transitive MOE. Let $\psi: G \longrightarrow H$ be a homomorphism. Then the composition $\psi cM: S \times S \longrightarrow H$ is a transitive MOE.

Proof: The fact that $\psi oM(x,y) = -\psi oM(y,x)$ follows from elementary group theory. Since $\psi oM(x,y) + \psi oM(y,z) = \psi(M(x,y) + M(y,z))$ and since $M(x,y) + M(y,z) = M(x,z)$ by transitivity, it is clear that $\psi oM(x,y) + \psi oM(y,z) = \psi oM(x,z)$.

The main result of this section is:

4.5 Theorem: Let $M_i: S \times S \longrightarrow G_i$ for $i = 1,\ldots n$, be transitive MOE's. Let $\psi_i: G_i \longrightarrow G$ be homomorphisms. Then there exists a homomorphism $\psi: \underset{i=1}{\overset{n}{X}} G_i \longrightarrow G$ such that $\psi o(\underset{i=1}{\overset{n}{X}} M_i): S \times S \longrightarrow G$ is a transitive MOE and such that for any k with $1 \leq k \leq n$ the following diagram is consistent:



where $I_k: G_k \longrightarrow \underset{i=1}{\overset{n}{X}} G_i$ is defined by $I_k(g_k) = (0, 0, \ldots g_k, \ldots 0)$.

Proof: Define $\psi = \sum_{i=1}^{n} \psi_i \circ P_i$, that is $\psi(g_1, g_2, \ldots g_n) = \sum_{i=1}^{n} \psi_i \circ P_i(g_1, g_2, \ldots g_n) = \sum_{i=1}^{n} \psi_i(g_i)$, where $P_k: \underset{i=1}{\overset{n}{X}} G_i \longrightarrow G_k$ is the kth projection defined by $P_k(g_1, g_2, \ldots g_n) = g_k$.

The consistency of the above diagram, and the fact that ψ is a homomorphism, are elementary results of group theory. Hence it is only necessary to verify that $\psi \circ (\underset{i=1}{\overset{n}{X}} M_i): S \times S \longrightarrow G$ is a transitive MOE. But $\psi \circ (\underset{i=1}{\overset{n}{X}} M_i)(x,y) = \psi(\underset{i=1}{\overset{n}{X}} M_i(x,y)) = \sum_{i=1}^{n} \psi_i(M_i(x,y)) =$ (by lemma 4.4) $\sum_{i=1}^{n} [-\psi_i(M_i(y,x))] = -[\sum_{i=1}^{n} \psi_i(M_i(y,x))] = -\psi \circ (\underset{i=1}{\overset{n}{X}} M_i)(y,x)$. Transitivity results from the following equalities:

$$\psi \circ (\underset{i=1}{\overset{n}{X}} M_i)(x,y) + \psi \circ (\underset{i=1}{\overset{n}{X}} M_i)(y,z) = \psi(\underset{i=1}{\overset{n}{X}} M_i(x,y)) + \psi(\underset{i=1}{\overset{n}{X}} M_i(y,z)) =$$

$$\psi(\underset{i=1}{\overset{n}{X}} (M_i(x,y) + M_i(y,z))) = \psi(\underset{i=1}{\overset{n}{X}} M_i)(x,z).$$

4.6 Corollary: The sum of transitive MOE's is a transitive MOE.

Proof: Merely take $G_i = G$ and $\psi_i = Id$ (the identity function) in theorem 4.5.

The preceeding theory must be applied with some caution. It might be inferred from corollary 4.6 that any two transitive MOE's could be added to produce a new transitive MOE. From a computational point of view, this is the case. However, the danger is that an attempt to add dissimilar quantities may produce meaningless results. Transitive MOE's which possess the same physical units may be added or otherwise combined, according to corollary 4.6. Transitive MOE's possessing dissimilar units may likewise be linearly combined, using theorem 4.5, but careful attention should be given to the units of the resulting MOE, so that the significance of the information is clearly understood.

Transitive MOE's are of central importance in the general theory of measures of effectiveness, due to the simplicity with which they may be combined to yield new transitive MOE's. Hence, it is desirable to have a method of deriving transitive MOE, given an arbitrary MOE.

Two methods will be discussed. The first one is somewhat unsatisfactory in the following sense: Suppose that y immediately succeeds x (y immediately succeeds x if $S_x \lesssim S_y$ and if $S_x \lesssim S_z$ for some $z \in S$, then $S_y \lesssim S_z$). Then the derived MOE M* does not necessarily preserve the values of successive pairs; i.e. there is no more than one successive pair $(x,y) \in S \times S$ such that $M(x,y) = M^*(x,y)$.

4.7 Theorem: Let $M: S \times S \longrightarrow G$ be a MOE. Suppose, for each $x \in S$, the sum $\sum_{u \leq x} M(x,u)$ is defined. Then the function $M^*: S \times S \longrightarrow G$ defined by $M^*(x,y) = \sum_{u \leq x} M(x,u) - \sum_{v \leq y} M(y,v)$ is a transitive MOE. Moreover, M* generates the same rank-ordering as M.

Proof: The first step is to show that $M^*(x,y) = -M^*(y,x)$. But

$$M^*(x,y) = \sum_{u \leq x} M(x,u) - \sum_{v \leq y} M(y,v) = -\left[\sum_{v \leq y} M(y,v) - \sum_{u \leq x} M(x,u) \right] = -M^*(y,x).$$

Next it is necessary to verify that $M^*(x,y) + M^*(y,z) = M^*(x,z)$ for all $x,y,z \in S$. But

$$M^*(x,y) + M^*(y,z) = \left[\sum_{u \leq x} M(x,u) - \sum_{v \leq y} M(y,v) \right] + \left[\sum_{w \leq y} M(y,w) - \sum_{t \leq z} M(z,t) \right] = \sum_{u \leq x} M(x,u) - \sum_{t \leq z} (z,t) = M^*(x,z).$$

Finally, it is necessary to check that $(S_x, M) = (S_x, M^*)$, and that $(S_x,M) < (S_y, M)$ if and only if $(S_x, M^*) < (S_y, M^*)$. Suppose $M(x,y) = 0$. In order to show that

$$\sum_{u \leq x} M(x,u) - \sum_{v \leq y} M(y,v) = 0$$

it is sufficient establish that $\{u: u \leq x\} = \{v: v \leq y\}$, since u and v are dummy variables. But this equality is an easy consequence of proposition 2.12. Hence, $\sum_{u \leq x} M(x,u) = \sum_{v \leq y} M(y,v)$, and the result follows:

Conversely, suppose that $M(x,y) \neq 0$. The objective is to show that $M^*(x,y) \neq 0$. Without loss of generality, assume that $M(x,y) < 0$. Then for $u \leq x < y$ it follows by definition that $M(u,y) \leq \min \{M(u,x), M(x,y)\} < 0$, so that $0 < M(x,u) \leq M(y,u)$ and $M(x,u) - M(y,u) \leq 0$. Hence,

$$\sum_{u \leq x} M(x,u) - \sum_{v \leq y} M(y,v) = \sum_{u < x} M(x,u) - \sum_{v < x} M(y,v) - \sum_{x \leq v \leq y} M(y,v).$$

Note that $\sum_{u < x} M(x,u) - \sum_{v < x} M(y,v) = \sum_{u < x} M(x,u) - \sum_{u < x} M(y,u) =$

$\sum_{u < x} (M(x,u) - M(y,u)) \leq 0$ by the above argument. Moreover,

$-\sum_{x \leq v \leq y} M(y,v) \leq -M(y,x)$ since, for $x \leq v \leq y$, $0 \leq M(y,v)$. As a result,

$$M^*(x,y) = \sum_{u < x} M(x,u) - \sum_{v < y} M(y,v) - \sum_{x \leq v \leq y} M(y,v) \leq -M(y,x) =$$

$M(x,y) < 0$. The argument is similar if $0 < M(x,y)$. In summary, $M(x,y) = 0$ if and only if $M^*(x,y) = 0$.

The preceeding argument proves even more: namely, $M(x,y) < 0$ if and only if $M^*(x,y) < 0$, so that $(S_x, M) = (S_x, M^*)$ and $(S_x, M) < (S_y, M^*)$, as was to be proved.

A second method of constructing a transitive MOE from an arbitrary MOE will now be discussed. This method has the advantage that it preserves some of the quantitative information of the original MOE, but the disadvantage that the set $\mathcal{S}$ of rank ordered equivalence classes of S must be well-ordered.

4.8 Definition: Let $(S, \leq)$ be a nonempty set with a partial order (see definition 1.2). Let $T \subset S$ be a nonempty subset. Then T is said to have a least element if $\exists x \in T$ such that $\forall y \in T$, $x \leq y$. (See 7 p. 53). It is easy to prove that if T has a least element, it is unique.

4.9 Definition: Let $(S, \leq)$ be a partially ordered set. S is said to be well-ordered if every nonempty subset possesses a least element. (See 7 p. 53).

4.10 Definition: Let $(S, \leq)$ be a well-ordered set. Let $x \in S$ and consider the set $\{y: x \leq y \text{ and } x \neq y\} = U_x$. If U_x is nonempty the least element x^+ of U_x will be called the successor of x.

4.11 Proposition: In any well-ordered set there is at most one element which does not have a successor.

Proof: Let S be well-ordered and suppose there are two elements $x,y \in S$ which do not have successors. Then $\{z: x \leq z \text{ and } x \neq z\} = \emptyset$ and $\{w: y \leq w \text{ and } y \neq w\} = \emptyset$, by definition. Consider the set $\{x,y\}$. Since S is well-ordered, $\{x,y\}$ has a least element, say x. Then $x \leq y$, so that $y \in \{z: x \leq z \text{ and } x \neq z\}$, which is impossible. Hence $x = y$.

4.12 Proposition: Every well-ordered set is linearly ordered.

Proof: Let S be well-ordered by a relation $\leq$. Pick $x,y \in S$. Consider $\{x,y\}$. By well-ordering, $\{x,y\}$ has a least element, say x. Then by definition $x \leq y$.

Proposition 4.12 establishes that the concept of well-ordering is stronger than the concept of linear ordering. The relationship of well-ordering to linear ordering is a proper one (i.e. the notions are not equivalent). For example, the rational numbers under the usual definition of $\leq$ are linearly ordered but not well-ordered.

4.13 Definition: Let S be a well-ordered set. A finite chain in S is a set $\{x_1, x_2, \ldots x_n\} \subset S$ such that x_{i+1} is the successor of x_i for $i = 1, \ldots n - 1$.

4.14 Theorem: Let M be a MOE on a set S, with associated induced partition $\mathcal{S}$. Suppose that $\mathcal{S}$ is well-ordered, and that any x, y ϵ S can be spanned by a finite chain, in the sense that $\exists\, x_0 = x, x_1, \ldots x_n = y$ with $x_k \epsilon (S_x)^k = ((S_x)^{k-1})^+$. Then $\exists$ a transitive MOE $M^\#$ with the following properties:

a) $M^\#$ generates the same rank order as M

b) For x ϵ S, $x^+ \epsilon (S_x)^+$, $M(x,x^+) = M^\#(x,x^+)$

c) $M^\#$ is unique

Proof: The proof is constructive in nature. Define a function $m: \mathcal{S} \longrightarrow G$ as follows:

Case 1: $S_x \epsilon \mathcal{S}$ has a successor, denoted $(S_x)^+$. Then $m(S_x) = M(x,x^+)$, where $x^+ \epsilon (S_x)^+$.

Case 2: $S_x \epsilon \mathcal{S}$ does not have a successor (see proposition 4.11). Then $m(S_x) = 0$. That m is well defined is a consequence of proposition 2.12. Note that $m(S_x) \leq 0$, since for x ϵ S_x, $x^+ \epsilon (S_x)^+$, $M(x,x^+) < 0$. Define $M^\#: S \times S \longrightarrow G$ via:

$$M^\#(x,y) = \sum_{S_x \lesssim S_u < S_y} m(S_u) \text{ if } x < y,\ M^\#(x,y) = 0 \text{ if } S_x = S_y,\ M^\#(x,y) = -\sum_{S_y \lesssim S_u < S_x} m(S_u) \text{ if } y < x$$

Step 1: Note that of M(x,y) and M(y,x) one quantity is positive or zero and the other is negative or zero. If M(x,y) = 0 then $x \equiv y$ so that $S_x = S_y$ and $M^\#(x,y) = -M^\#(y,x) = 0$. If M(x,y) $\neq$ 0, suppose without loss of generality that M(x,y) < 0. Then $M^\#(x,y) = \sum_{S_x \lesssim S_u < S_y} m(S_u)$. On the other hand, for

$y < x$, $M^{\#}(y,x) = -\sum_{S_x \lesssim S_u < S_y} m(S_u)$ so that $M^{\#}(x,y) = -M^{\#}(y,x)$.

Step 2: The proof of transitivity will be in three parts.

Case 1: $M^{\#}(y,z) = 0$. Then $S_y = S_z$ so that $M^{\#}(x,y) + M^{\#}(y,z) = \sum_{S_x \lesssim S_u < S_y} m(S_u) = \sum_{S_x \lesssim S_u < S_z} m(S_u) = M^{\#}(x,z)$

Case 2: $M^{\#}(x,y)$ and $M^{\#}(y,z)$ are both positive or both negative. Say $M^{\#}(x,y) < 0$ and $M^{\#}(y,z) < 0$. Then $M^{\#}(x,y) + M^{\#}(y,z) = \sum_{S_x \lesssim S_u < S_y} m(S_u) + \sum_{S_y \lesssim S_u < S_z} m(S_u) = \sum_{S_x \lesssim S_u < S_z} m(S_u) = M^{\#}(x,z)$. The proof is similar if $M^{\#}(x,y)$ and $M^{\#}(y,z)$ are positive.

Case 3: $M^{\#}(x,y)$ and $M^{\#}(y,z)$ have different signs. Say $M^{\#}(x,y) < 0$ and $0 < M^{\#}(y,z)$. Then $M^{\#}(x,y) + M^{\#}(y,z) = \sum_{S_x \lesssim S_u < S_y} m(S_u) - \sum_{S_z \lesssim S_u < S_y} m(S_u) =$

$$\begin{cases} \sum_{S_x \lesssim S_u < S_z} m(S_u) & \text{if } x < z \\ 0 & \text{if } x \equiv z \\ -\sum_{S_z \lesssim S_u < S_x} m(S_u) & \text{if } z < x \end{cases}$$

Thus if $x < z$, $\sum_{S_x \lesssim S_u < S_z} m(S_u) = M^{\#}(x,z)$. Likewise if $z < x$, $-\sum_{S_z \lesssim S_u < S_x} M(S_u)$

$-M^{\#}(z,x) = M^{\#}(x,z)$.

Step 3. It is necessary to show that $(S_x, M) = (S_x, M^{\#})$ for $x \in S$, and that $(S_x, M) < (S_y, M)$ if and only if $(S_x, M^{\#}) < (S_y, M^{\#})$. Both results are consequences of the definition of $M^{\#}$; $M^{\#}(x,y) = 0$ if and only if $M(x,y) = 0$; likewise $M^{\#}(x,y) < 0$ if $M(x,y) < 0$ and $0 < M^{\#}(x,y)$ if $0 < M(x,y)$.

Step 4: The next step is to verify part b) of the theorem; namely, $M(x,x^+) = M^{\#}(x,x^+)$ for $x \in S_x$, $x^+ \in (S_x)^+$. But $M^{\#}(x,x^+) = \sum_{S_x \lesssim S_u < S_{x^+}} m(S_u) =$

$\sum_{S_x \lesssim S_u < (S_x)^+} m(S_u) = m(S_x) = M(x,x^+)$, since $(S_x)^+ = S_{x^+}$ is the successor of S_x.

Step 5: Uniqueness follows from the definition of M and the fact that M can be decomposed as a sum of values obtained by pairing elements of successive equivalence classes.

4.15 Conjecture: Let M and S satisfy the hypotheses of theorem 4.14. Then the set S is at most countable.

MEASURES OF CORRELATION

This section will have a distinctly geometric character. Several methods of measuring the correlation and/or proximity of two measures of effectiveness will be discussed. As indicated in footnote 1, page 18, a measure of effectiveness can be visualized in matrix form; and hence as a vector. Thus under suitable hypotheses a substantial portion of the theory of elementary geometry and algebra can be applied to provide insight into the relationship between two measures of effectiveness. A few preliminary definitions and lemmas are needed. Throughout this section it will be assumed that the linear O-groups are all fields, and hence isomorphic to subfields of the real numbers.

5.1 Observation: Let M: $S \times S \longrightarrow F$ be a MOE. Since F is a field, M can be viewed as an element of the vector space $V_{S \times S} = \{f: S \times S \longrightarrow F\}$ where the addition and scaling of vectors is according to the usual rules: $(f + g)(x,y) = f(x,y) + g(x,y)$, and $\lambda(f(x,y)) = \lambda f(x,y)$.

Observation 5.1 provides the proper setting for the discussion to follow. The related notions of inner product and distance will be investigated abstractly, and an application will be made to MOE.

5.2 Definition: Let V be a vector space. An inner product on V is a function $\emptyset$: $V \times V \longrightarrow F$, F an ordered field in which every positive element has a square root, such that:

a) $\emptyset$ is bilinear

b) $\emptyset$ is symmetric

c) $\emptyset$ is positive definite, $\emptyset(x,x) \geq 0$

d) $\emptyset(x,x) = 0$ if and only if $x = 0$

Generally speaking, the inner product of two vectors x,y is denoted $x \cdot y$ rather than $\emptyset(x,y)$. Some simple properties of inner products will be reviewed.

5.3 Definition: Let V be a vector space with an inner product. The length of a vector $x \in V$ is defined to be: $\|x\| = \sqrt{x \cdot x}$. The distance d(x,y) between two vectors is: $d(x,y) = \|y - x\| = \sqrt{(y-x) \cdot (y-x)}$.

5.4 Lemma: Let $x, y \in V$. Then the following inequality holds:

$$\min \{x \cdot x, y \cdot y\} \leq \|x\| \, \|y\| \leq \max \{x \cdot x, y \cdot y\}.$$

Proof: This is a trivial drill in elementary field theory. Note that $x \cdot x = \|x\|^2$ and $y \cdot y = \|y\|^2$. Suppose for example that $\|x\| \leq \|y\|$. Then $\|x\|^2 \leq \|x\| \, \|y\|$ and similarly $\|x\| \, \|y\| \leq \|y\|^2$.

5.5 Lemma: Let $x,y \in V$, $x,y \neq 0$, V an inner product space. Then

$$\frac{|x \cdot y|}{\max \{x \cdot x, \, y \cdot y\}} \leq \frac{|x \cdot y|}{\|x\| \, \|y\|} \leq \frac{|x \cdot y|}{\min \{x \cdot x, \, y \cdot y\}}$$

Proof: Using lemma 5.4, one deduces that

$$\frac{1}{\max \{x \cdot x, \, y \cdot y\}} \leq \frac{1}{\|x\| \, \|y\|} \leq \frac{1}{\min \{x \cdot x, \, y \cdot y\}}$$

Since $0 \leq |x \cdot y|$, the result follows by multiplying each element of the inequality by $|x \cdot y|$.

5.6 Lemma: If V is an inner product space and $x, y \in V$ are nonzero, then $-1 \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1$

Proof: Lemma 5.6 is a direct consequence of the well-known Schwarz inequality $|x \cdot y| \leq \|x\| \|y\|$ (see 2 p. 3). Thus $\frac{|x \cdot y|}{\|x\| \|y\|} \leq$ or equivalently $-1 \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1$.

An observation is in order at this point. The quantity $\frac{x \cdot y}{\|x\| \|y\|}$ is called the cosine of the angle between the vectors x and y. Under a suitable inner product, $\frac{x \cdot y}{\|x\| \|y\|}$ reduces to the notion of the cosine as defined in elementary trigonometry.

5.7 Lemma: Let x, y be nonzero vectors of an inner product space V. Then $\frac{x \cdot y}{\max\{x \cdot x, y \cdot y\}} = 1$ if and only if $x = y$.

Proof: If $x = y$, then clearly $x \cdot x = y \cdot y = x \cdot y$, so that $\frac{x \cdot y}{\max\{x \cdot x, y \cdot y\}} = \frac{x \cdot y}{x \cdot y} = 1$. Conversely, suppose that $\frac{x \cdot y}{\max\{x \cdot x, y \cdot y\}} = 1$. Clearly $x \cdot y > 0$, since $x \cdot x$ and $y \cdot y$ are positive. Thus by lemmas 5.5 and 5.6, $1 = \frac{x \cdot y}{\max\{x \cdot x, y \cdot y\}} \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1$ so that $\|x\| \|y\| = \sqrt{(x \cdot x)(y \cdot y)} = \max\{x \cdot x, y \cdot y\}$. Assume without loss of generality that $y \cdot y \leq x \cdot x$. Then $(x \cdot x)(y \cdot y) = (x \cdot x)^2$, or $x \cdot x = y \cdot y$, so that $x \cdot y = \|x\| \|y\| = x \cdot x = y \cdot y$. One obtains $x \cdot (x - y) = 0$ and $y \cdot (x - y) = 0$; thus $(x - y) \cdot (x - y) = x \cdot (x - y) - y \cdot (x - y) = 0$. Since $(x - y) \cdot (x - y) = 0$ if and only if $x - y = 0$, it follows that $x = y$.

5.8 Lemma: Let $x, y \in V$ be as in lemma 5.7. Then $\frac{|x \cdot y|}{\|x\| \|y\|} = 1$ if and only if there exists a scalar $\lambda \neq 0$ such that $x = \lambda y$.

Proof: This is a well known consequence of the Schwarz inequality (see 2 p. 4).

The discussion now returns to measures of effectiveness. A preliminary definition is needed.

5.9 Definition: Let $M: S \times S \longrightarrow F$ be a MOE. The associated uniform MOE $\overline{M}$ is defined as follows:

$$\overline{M}(x,y) = \begin{cases} \dfrac{M(x,y)}{|M(x,y)|} & \text{if } M(x,y) \neq 0 \\ 0 & \text{if } M(x,y) = 0 \end{cases}$$

It is straightforward to prove that $\overline{M}$ is a MOE and that it induces the same rank ordering on S as does M.

Since the set $V_{S \times S}$ of observation 5.1 can be viewed as a vector space in a natural way, it is useful to define an inner product on it. For $f, g \in V_{S \times S}$, define f*g by: $f*g = \sum_{S \times S} f(x,y)\, g(x,y)$, provided the sum exists. It is easy to verify, assuming that f*g exists for all $f, g \in V_{S \times S}$, that the operation * is an inner product. Note that $\| f \| = \sqrt{f*f}$ and that $d(f,g) = \| g - f \|$. In particular, the above definition applies if f and g are MOE's. Four more definitions can now be made.

5.10 Definition: Let M, N: $S \times S \longrightarrow F$ be MOE's. The distance between M and N is $d(M,N) = \|N - M\| = \sqrt{\sum_{S \times S} (N(x,y) - M(x,y))^2}$.

Viewing M and N as vectors, d(M,N) gives a measure of the proximity of the two MOE.

5.11 Definition: The quantity $\dfrac{M*N}{\max\{M*M,\ N*N\}}$ will be called the strong correlation (denoted S - corr(M,N).

5.12 Definition: $\frac{M*N}{\|M\|\ \|N\|}$ is the correlation between M and N, denoted corr (M,N). Observe that corr (M,N) resembles the statistical concept of the correlation coefficient, as will be explained below.

5.13 Definition: The concept of weak correlation uses definition 5.9. The quantity W-corr $(M,N) = \frac{\overline{M} * \overline{N}}{\min\{\overline{M}*\overline{M},\ \overline{N}*\overline{N}\}}$ is defined to be the weak correlation between M and N. Notice that W-corr is in a sense a normalized quantity.

5.14 Observation: The preceeding definitions (especially definition 5.12) have a distinct statistical flavor. Recall that for random variables X and Y with means $\mu_1 = E(X)$ and $\mu_2 = E(Y)$, and variances $\sigma_1^2 = E((X - \mu_1)^2)$ and $\sigma_2^2 = E((X - \mu_2)^2)$, the correlation coefficient ρ_{12} is defined by:

$$\rho_{12} = \frac{E((X - \mu_1)(Y - \mu_2))}{\sigma_1\ \sigma_2}$$

Suppose that $\mu_1 = \mu_2 = 0$. Then $\rho_{12} = \frac{E(XY)}{\sqrt{E(X^2)}\ \sqrt{E(Y^2)}}$. Notice the similarity between ρ_{12} with $\mu_1 = \mu_2 = 0$ and the correlation $\frac{M*N}{\|M\|\|N\|}$ of definition 5.12. In fact, the similarity between the two expressions is more than formal if the cardinality of the object set S of the MOE's M and N is finite (say K). Note that $\sum_{S \times S} M(x,y) = \sum_{S \times S} N(x,y) = 0 \Longrightarrow \mu_1 = \mu_2 = 0$ in this case.[1] Defining $K^2\ E(MN) = M*N$, $K^2\ E(M^2) = M*M$, and $K^2\ E(N^2) = N*N$, one sees that $\frac{E(MN)}{\sqrt{E(M^2)}\ \sqrt{E(N^2)}} = \frac{M*N}{\|M\|\ \|N\|}$. (see 3 p. 69).

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[1] Due to skew symmetry.

5.15 Theorem: Given two MOE's M, N: $S \times S \longrightarrow F$, S-corr(M,N) = 1 if and only if M = N.

Proof: A direct consequence of lemma 5.7.

5.16 Theorem: With the hypothesis of theorem 5.15, it follows that corr (M,N) = 1 if and only if there is a nonzero scalar k ϵ f such that M = kN. The proof is contained in any good book on vector analysis. See for example (2, p. 4). Theorem 5.15 gives necessary and sufficient conditions for two MOE's to be equal, and theorem 5.16 gives similar conditions for two MOE's to be proportional. These theorems suggest that the amount by which S-corr and corr deviate from 1 are measures of the deviation from proportionality and equality of two MOE's, by a continuity argument.

5.17 Theorem: Under the hypotheses of theorem 5.15, W-corr(M,N) = 1 if and only if the ranking induced by N is consistent with the ranking induced by M, or vice versa (see proposition 3.8).

Proof: Unfortunately theorem 5.17 does not admit a neat geometrical proof such as those of theorems 5.15 and 5.16. Rather, a computational technique will be employed.

Suppose first that N is consistent with M. Consider $\overline{M}$ and $\overline{N}$. By construction, the values of $\overline{M}$ and $\overline{N}$ are either 1, 0, or -1. Since $\overline{M}(x,y) < 0 \Longrightarrow \overline{N}(x,y) \leq 0$ and $\overline{M}(x,y) = 0 \Longrightarrow \overline{N}(x,y) = 0$ it follows that $\overline{N}(x,y) = 0$ or -1 if $\overline{M}(x,y) = -1$, and $\overline{N}(x,y) = 0$ if $\overline{M}(x,y) = 0$. Similarly $\overline{M}(x,y) = 1 \Longrightarrow \overline{N}(x,y) = 0$ or 1. As a result, $\overline{N}^2(x,y) = \overline{N}(x,y) \cdot \overline{N}(x,y) = 1 \Longrightarrow \overline{M}^2(x,y) = 1$ (note that $\overline{M}^2(x,y) = 0$ or 1 and $\overline{N}^2(x,y) = 0$ or 1), so that $\sum_{S \times S} \overline{N}^2(x,y) \leq \sum_{S \times S} \overline{M}^2(x,y)$. On the other hand, $\sum_{S \times S} \overline{M}(x,y)\, \overline{N}(x,y) =$

$= \sum_{U} \overline{M}(x,y)\,\overline{N}(x,y)$, where $U = \{(x,y) \in S \times S: \overline{N}(x,y) \neq 0\}$. But for $(x,y) \in U$, $\overline{M}(x,y) \neq 0$. If $\overline{M}(x,y) = -1$, $\overline{N}(x,y) = -1$ while $\overline{M}(x,y) = 1 \Longrightarrow \overline{N}(x,y) = 1$. In short, $\overline{M}(x,y) = \overline{N}(x,y)$, for $(x,y) \in U$. Then $\sum_{U} \overline{M}(x,y)\,\overline{N}(x,y) = \sum_{U} \overline{N}^2(x,y)$ $= \sum_{S \times S} \overline{N}^2(x,y)$. The conclusion is that:

$$\frac{\sum_{S \times S} \overline{M}(x,y)\,\overline{N}(x,y)}{\min\{\sum_{S \times S} \overline{M}^2(x,y),\ \sum_{S \times S} \overline{N}^2(x,y)\}} = \frac{\sum_{S \times S} \overline{M}(x,y)\,\overline{N}(x,y)}{\sum_{S \times S} \overline{N}^2(x,y)} = \frac{\overline{M} * \overline{N}}{\min\{\overline{M} * \overline{M},\ \overline{N} * \overline{N}\}} = 1.$$

To prove the converse, suppose that $\overline{N} * \overline{N} \leq \overline{M} * \overline{M}$, and that $\frac{\overline{M} * \overline{N}}{\overline{N} * \overline{N}} = 1$ Then $\overline{M} * \overline{N} = \overline{N} * \overline{N}$, or $\overline{N} * (\overline{M} - \overline{N}) = 0$. The objective is to show that $\overline{M}(x,y) = 0 \Longrightarrow \overline{N}(x,y) = 0$ and that $\overline{M}(x,y) = -1 \Rightarrow \overline{N}(x,y) = 0$ or -1. The discussion will proceed by cases.

Case 1: Suppose $\overline{N}(x,y) = 1$. Then by computation,

$$\left.\begin{array}{l}\overline{M}(x,y) = 1\\ \overline{M}(x,y) = 0\\ \overline{M}(x,y) = -1\end{array}\right\} \Longrightarrow \overline{N}(x,y)\,(\overline{M}(x,y) - \overline{M}(x,y)) = \begin{cases}0\\-1\\-2\end{cases}$$

Case 2: $\overline{N}(x,y) = 0$. No information can be obtained about $\overline{M}(x,y)$, because $\overline{N}(x,y)\,(\overline{M}(x,y) - \overline{N}(x,y)) = 0$ in any case.

Case 3: $\overline{N}(x,y) = -1$. Note that:

$$\left.\begin{array}{l}\overline{M}(x,y) = 1\\ \overline{M}(x,y) = 0\\ \overline{M}(x,y) = -1\end{array}\right\} \Longrightarrow \overline{N}(x,y)\,(\overline{M}(x,y) - \overline{N}(x,y)) = \begin{cases}-2\\-1\\0\end{cases}$$

The above calculations show that $\overline{N}(x,y)\,(\overline{M}(x,y) - \overline{N}(x,y)) \leq 0$ regardless of the values of $\overline{M}(x,y)$ and $\overline{N}(x,y)$. However, by hypotheses $\overline{N} * (\overline{M} - \overline{N}) = \sum_{S \times S} \overline{N}(x,y)\,(\overline{M}(x,y) - \overline{N}(x,y)) = 0$. The only way this situation

can occur is if every individual component $\overline{N}(x,y)$ $(\overline{M}(x,y) - \overline{N}(x,y)) = 0$. It then follows that $\overline{N}(x,y) = 1 \Longrightarrow \overline{M}(x,y) = 1$ and $\overline{N}(x,y) = -1 \Longrightarrow \overline{M}(x,y) = -1$. These statements imply, in turn, that $\overline{N}(x,y) \neq 0 \Rightarrow \overline{M}(x,y) \neq 0$. By contraposition, $\overline{M}(x,y) = 0 \Rightarrow \overline{N}(x,y) = 0$. Suppose that $\overline{M}(x,y) = -1$. Using cases 1, 2, and 3 one sees that $\overline{N}(x,y)$ must be 0 or -1 in order for $\overline{N}(x,y)$ $(\overline{M}(x,y) - \overline{N}(x,y))$ to be zero.

The preceeding results give routine computational methods of calculating the consistency (or lack thereof) of two measures of effectiveness.

For example, it can now be verified computationally that the transitive measures of effectiveness derived in theorems 4.6 and 4.14 are strongly correlated with their respective generating MOE's.

EXAMPLES OF MEASURES OF EFFECTIVENESS

This section will be somewhat more sketchy than the preceeding ones. Some of the examples to be discussed have been introduced earlier. Others will follows easily from previous definitions and theorems.

6.1 Example: The Canonical MOE. A measure of effectiveness satisfying definition 2.4, with the additional properties that the linear O-group is an ordered field, and that $M(x,y) \in \{-1, 0, 1\}$ will be called a canonical MOE. An illustrative example would be helpful. Say that $S = \{A, B, C\}$; thus three objects are to be compared. Suppose that A is preferred to B, and B is preferred to C. Thus $C \lesssim B \lesssim A$. This information can be translated into a measure of effectiveness as follows: $M(C,C) = M(B,B) = M(A,A) = 0$, $M(C,B) = -1$ $M(B,A) = -1$, $M(B,C) = 1$, $M(A,B) = 1$, $M(C,A) = -1$, $M(A,C) = 1$. In matrix form, M looks like:

$$M = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \end{array}$$

The matrix of M is in canonical form if the rows are arranged according to the descending order of the induced ranking and the colums are arranged left to right according to the same scheme. Note that the measure M of this example consists of + 1's, 0's, and -1's. Actually the canonical MOE was first introduced in definition 5.9 as the uniform MOE derived from a given MOE.

The next topic to be treated is that of experimentally derived measures of effectiveness. Assume that S is a set of objects or alternatives to be compared. Assume that an experiment is devised which tests each alternative and assigns a number to the result. It will be seen that the experiment can be used to obtain a measure of effectiveness. In order to derive measures of effectiveness from experiments, it is necessary to have a formal definition of the notion of an experiment. The definition presented below is quite general in form, and thus has numerous applications.

6.2 Definition: Let S be an arbitrary nonempty set. Let G be a linear O-group. A function f: S $\longrightarrow$ G is an experiment on S. Generally G will be the real numbers under addition or the positive reals under multiplication.

A word of caution is appropriate at this point. Definition 1.3 may not be totally general in the sense that the result of an actual experiment may be a sequence of numbers, a vector, a complex number, etc.

However, for purposes of the discussion to follow it is important that the outcome of an experiment be expressed in terms of elements of a linear O-group. To cover the case in which the outcome of a physical experiment is a sequence, a vector, or a function, another definition is necessary.

6.3 Definition: Let G^I represent the set of all functions from a nonempty set I to a linear O-group G. That is, $G^I = \{f: I \longrightarrow G\}$. Let S be a nonempty set and $\emptyset: S \longrightarrow G^I$ be a function. Then $\emptyset$ will be called a sequence of experiments indexed on I.

6.4 Example: Experimentally Derived Measures of Effectiveness. Let $f: S \longrightarrow G$ be an experiment on S. Define a function $M: S \times S \longrightarrow G$ by $M(x,y) = f(x) - f(y)$. It is easy to check that M is a transitive measure of effectiveness.

Conversely, suppose that $M: S \times S \longrightarrow G$ is a transitive MOE. Then an experiment $f: S \longrightarrow G$ can be obtained from M with the property that $M(x,y) = f(x) - f(y)$. That is, one can recover the initial MOE by taking differences of experiment values. The experiment f is obtained by selecting $z \in S$ and defining $f(x) = M(x,z)$. Note that $f(x) - f(y) = M(x,z) - M(y,z) = M(x,y)$ by transitivity. Note also that f is unique to within a constant. That is, for $x, w \in S$ if $f(x) = M(x,z)$ and $g(x) = M(x,w)$ then $f(x) + M(z,s) = g(x)$.

Two specific applications of example 6.4 are readily available; namely take $G = (\underline{R}, +)$, the additive real numbers and $G = (\underline{R}^+, \cdot)$, the postive multiplicative reals. In the first case a MOE is a difference of experimental values. In the second case it is a quotient of experimental values. Most of the common measures of effectiveness, including the so called "ratio MOE's" are of the first type.

In the spirit of example 6.4 one can define a sequence of measures of effectiveness; given a sequence $\emptyset$: $S \longrightarrow G^I$ of experiments define $\underset{\alpha \in I}{X} M_\alpha$: $S \times S \longrightarrow G^I$ coordinatewise as follows: $M_\alpha(x,y) = P_\alpha \circ \emptyset(x) - P_\alpha \circ \emptyset(y)$, where $P_\alpha = G^I \longrightarrow G$ is the projection on the αth co-ordinate, define by $P_\alpha(g) = g_\alpha$. Then form the direct product $\underset{\alpha}{X} M_\alpha$. Addition is taken coordinatewise. Although $\underset{\alpha}{X} M_\alpha$ is not a measure of effectiveness, it can be used to form a mesure of effectiveness. Merely take $M(x,y) = \underset{\alpha}{\Sigma} M_\alpha(x,y)$, provide the sum exists for all $(x,y) \in S \times S$. $M(x,y)$ is a transitive measure of effectiveness by corollary 4.6.

6.5 Example: The Likelihood Ratio Test. Let $f(x \mid \theta)$ be the probability density function of a distribution with parameter $\theta \in \Omega$. Let $x_1, x_2, \ldots x_n$ be a random sample of size n. Assuming that the variables of the joint p.d.f. are independent, define $L(x_1, \ldots x_n \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$.

Let a_θ, $a_\emptyset$ represent the parametric outcomes associated with the choice of the values θ and $\emptyset$ respectively. Define a function λ as follows:

$$\lambda(\theta, \emptyset) = \frac{L(x_1, \ldots x_n \mid \theta)}{L(x_1, \ldots x_n \mid \emptyset)}$$

Suppose there exists a number k such that $\lambda(\theta,\emptyset) < k$ leads to the choice of a_θ, $\lambda(\theta,\emptyset) = 0$ yields no information, and $k < \lambda(\theta,\emptyset)$ leads to the choice of a_θ as the desired outcome. (Observe that necessarily $0 < k$.) The function $\lambda(\theta,\emptyset)$ is called the likelihood ratio. Using example 6.4 one radily verifies that $\frac{1}{k} \lambda(\theta,\emptyset)$ is a measure of effectiveness. Thus the likelihood ratio is essentially a measure of effectiveness. (See 3, p. 254 and 1, p. 248).

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A GENERALIZATION OF MINIMUM BIAS ESTIMATION FOR WEIGHTED LEAST SQUARES

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1. Introduction.

For response surface analysis, Karson, Manson and Hader [5] introduced an estimation procedure called minimum bias estimation, for estimating the parameters in the model equation. (Karson [4] later introduced a design criterion to provide protection against certain higher order models.) The minimum bias estimation procedure achieves minimum average squared bias of the fitted model without depending on the values of the unknown parameters of the true model. Also, provided only that the design satisfies a simple estimability condition and subject to providing minimum average squared bias, the minimum bias estimator also provides minimum average variance of $\hat{y}(\underset{\sim}{x})$ where $\hat{y}(\underset{\sim}{x})$ is the estimate of the response at some point $\underset{\sim}{x}$.

We shall extend the method of minimum bias estimation to include weighted least squares. An illustration is given comparing the method of minimum bias estimation to the approach of Box and Draper [1,2] for minimizing the mean square error of $\hat{y}(\underset{\sim}{x})$ averaged over some region R.

2. Development

Let $y(\underset{\sim}{x})$ be the observed response at a point $\underset{\sim}{x} = (x_1, x_2, \ldots, x_k)$ in a restricted region of interest, say R, which forms part of the k-dimensional space. The true polynomial model (that is, the model which exactly fits the response) over the region R is,

$$E(y(\underset{\sim}{x})) = \eta(\underset{\sim}{x}) = \underset{\sim}{x}_1'\underset{\sim}{\beta}_1 + \underset{\sim}{x}_2'\underset{\sim}{\beta}_2 \quad , \tag{2.1}$$

where $\eta(\underset{\sim}{x})$ is of degree d_2. The experimenter however wishes to use a polynomial of degree d_1 (where d_1 is of lesser degree than d_2) of the form,

$$\hat{y}(\underset{\sim}{x}) = \underset{\sim}{x}_1'\underset{\sim}{b}_1 \quad , \tag{2.2}$$

to estimate or predict the response over the region of interest R. The elements of the vector $\underset{\sim}{x}_1$ make up the terms required for the polynomial of degree d_1 whereas the elements of the vector $\underset{\sim}{x}_2$ consist of the additional higher - degree terms required for the polynomial of degree d_2. The elements of the vectors $\underset{\sim}{\beta}_1$ and $\underset{\sim}{\beta}_2$ are unknown parameters associated with the elements in $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ respectively. Over N observed responses $y_i = \eta(x_i) + \epsilon_i$, $i = 1,2,\ldots,N$, at different points $\underset{\sim}{x}$ in R, we shall make the following assumptions about the N errors,

$$E(\epsilon_i) = 0, \quad E(\epsilon_i \epsilon_j) = \sigma^2/w_{ij} \qquad i,j=1,2,\ldots,N \tag{2.3}$$

where either the w_{ij} are known or ratios of the $w_{ij}/w_{ij'}$ $(j \neq j')$ are assumed known.

Karson [4], and Karson, Manson and Hader [5] discuss the minimization of the mean square error of $\hat{y}(\underset{\sim}{x})$, integrated over the region of interest R. The integrated mean square error of $\hat{y}(\underset{\sim}{x})$ is expressed as,

$$\frac{N\Omega}{\sigma^2} \int_R E\{\hat{y}(\underset{\sim}{x}) - \eta(\underset{\sim}{x})\}^2 \, d\underset{\sim}{x}, \text{ where } \Omega^{-1} = \int_R d\underset{\sim}{x}. \tag{2.4}$$

The integrated mean square error of $\hat{y}(\underset{\sim}{x})$ can be separated into two components,

$$\frac{N\Omega}{\sigma^2}\int_R \{E(\hat{y}(\underset{\sim}{x})) - \eta(\underset{\sim}{x})\}^2 \, d\underset{\sim}{x} + \frac{N\Omega}{\sigma^2}\int_R E\{\hat{y}(\underset{\sim}{x}) - E(\hat{y}(\underset{\sim}{x}))\}^2 \, d\underset{\sim}{x} = B+V, \quad (2.5)$$

where B is the average squared bias of the model and V is the average variance of $\hat{y}(\underset{\sim}{x})$. In assuming $w_{ij} = 1 \quad i=j, \quad w_{ij} = 0 \quad i\neq j$ in (2.3), the authors [4,5] found that B is minimized over R if one fits the full model (2.1) to the N observations and takes $\hat{y}(\underset{\sim}{x}) = \underset{\sim}{x}_1'\underset{\sim}{b}_1$ where $\underset{\sim}{b}_1 = \underset{\sim}{A}\underset{\sim}{\beta} = \underset{\sim}{A}(\underset{\sim}{\beta}_1, \underset{\sim}{\beta}_2)'$ and $\underset{\sim}{A}$ is a moment matrix defined by,

$$\underset{\sim}{A} = [\underset{\sim}{I} \vdots \underset{\sim}{\mu}_{11}^{-1}\underset{\sim}{\mu}_{12}] \quad (2.6)$$

In other words in using the lesser degree model to estimate the response, one uses a model in which the parameter estimates $\underset{\sim}{b}_1$ are linear functions of the estimates of the parameters in the true model of higher degree. The matrices $\underset{\sim}{\mu}_{ij}$ defined by,

$$\underset{\sim}{\mu}_{ij} = \Omega\int_R \underset{\sim}{x}_i\underset{\sim}{x}_j \, d\underset{\sim}{x}, \quad i\leq j=1,2 \quad (2.7)$$

are called region moment matrices and $\underset{\sim}{I}$ is an identity matrix.

Now, when the assumptions are as stated in (2.3), the minimum value of B is,

$$\text{Min } B = \frac{N}{\sigma^2}\underset{\sim}{\beta}_2'\{\underset{\sim}{\mu}_{22} - \underset{\sim}{\mu}_{12}'\underset{\sim}{\mu}_{11}^{-1}\underset{\sim}{\mu}_{12}\}\underset{\sim}{\beta}_2 \quad (2.8)$$

and Min B is attained when

$$E(\underset{\sim}{b}_1) = \underset{\sim}{A}\underset{\sim}{\beta}. \quad (2.9)$$

If we write $\underset{\sim}{b}_1$ as a linear combination of the observations,

$$\underset{\sim}{b}_1 = \underset{\sim}{L}\underset{\sim}{y},$$

and note that $E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{\beta}$ where $\underset{\sim}{X} = (\underset{\sim}{X}_1\underset{\sim}{X}_2)$ and $\underset{\sim}{X}_i$ is the matrix of values taken by the terms in $\underset{\sim}{x}_1'$ in (2.1) over the N experimental combinations, then the matrix $\underset{\sim}{L}$ must satisfy $\underset{\sim}{L}\underset{\sim}{X} = \underset{\sim}{A}$ from (2.9).

For the class of designs satisfying the condition,

$$(\underset{\sim}{X}_1'\underset{\sim}{W}^{-1}\underset{\sim}{X}_1)^{-1}\ \underset{\sim}{X}_1'\underset{\sim}{W}^{-1}\underset{\sim}{X}_2 = \underset{\sim}{\mu}_{11}^{-1}\underset{\sim}{\mu}_{12}\quad , \tag{2.11}$$

where $\underset{\sim}{W} = [\frac{1}{w_{ij}}]$ is the N x N covariance matrix of the errors, the standard weighted least squares estimator using,

$$\underset{\sim}{L} = (\underset{\sim}{X}_1'\underset{\sim}{W}^{-1}\underset{\sim}{X}_1)^{-1}\ \underset{\sim}{X}_1'\underset{\sim}{W}^{-1}\quad , \tag{2.11}$$

does satisfy $\underset{\sim}{L}\underset{\sim}{X} = \underset{\sim}{A}$. [Note, in using the expression (2.12) for $\underset{\sim}{L}$ to obtain the estimates $\underset{\sim}{b}_1$ in (2.10), this is exactly the weighted least squares formula; see for example [3], p. 77-81]. As pointed out in [5] where $\underset{\sim}{W} = \underset{\sim}{I}$, (2.12) is the Box and Draper result.

The average variance V expressed as,

$$V = \frac{N\Omega}{\sigma^2}\int_R \text{Var}\ \hat{y}(\underset{\sim}{x})\ d\underset{\sim}{x}, \tag{2.13}$$

is minimized when $\hat{y}(\underset{\sim}{x})$ is the weighted least squares estimate of its expectation at all points in R. Since $E[\hat{y}(\underset{\sim}{x})] = \underset{\sim}{x}_1'\ \underset{\sim}{A}\underset{\sim}{\beta}$, then V is minimized when $\underset{\sim}{x}_1'\underset{\sim}{b}_1 = \underset{\sim}{x}_1'\underset{\sim}{A}(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{y}$, that is, when $\underset{\sim}{L}$ is,

$$\underset{\sim}{L} = \underset{\sim}{A}(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{W}^{-1}\quad . \tag{2.14}$$

Since the variance of the estimate $\hat{y}(\underset{\sim}{x})$ is,

$$\text{Var } \hat{y}(\underset{\sim}{x}) = \underset{\sim}{x}_1'\underset{\sim}{A}(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{A}'\underset{\sim}{x}_1\sigma^2 , \qquad (2.15)$$

then the expression for V is,

$$V = N \text{ trace } \{\underset{\sim}{A}(\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{A}'\underset{\sim}{\mu}_{11}\} . \qquad (2.16)$$

Before we discuss an example, let us consider briefly the effect of the error structure $E(\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}') = \underset{\sim}{W}\sigma^2$ on the contributions B and V. From (2.8) and (2.16), one observes that only the average variance V is affected by considering $\underset{\sim}{W} \neq \underset{\sim}{I}$. The minimum value of average squared bias depends only on the region moment matrices $\underset{\sim}{\mu}_{ij}$ and the size of the elements of the vector $\underset{\sim}{\beta}_2$ which appear in the true model but are ignored in the fitted model. That is, although the estimates of the parameters in the fitted model may be biased, the amount of bias is not affected by $w_{ij} \neq 1$.

We now discuss example (a) in [5] using assumptions (2.3).

3. An Example

Let the true polynomial and prediction equation respectively be

$$\eta(\underset{\sim}{x}) = \beta_0 + \beta_1 x + \beta_2 x^2 \quad \text{and} \quad \hat{y}(\underset{\sim}{x}) = \underset{\sim}{x}_1'\underset{\sim}{b}_1 = b_0 + b_1 x. \qquad (3.1)$$

Consider n_ℓ experiments performed at each of the settings $x = \pm\ell$ and $n_0 \geq 1$ experiments at $x = 0$. Further assume that at $x = \pm\ell$, $\text{Var}(y_i) = \sigma_\ell^2 = \sigma^2/w_\ell$ and at $x = 0$, $\text{Var}(y_i) = \sigma_0^2 = \sigma^2/w_0$, w_0 could be different from w_ℓ, and that the observations are uncorrelated.

Then $\underset{\sim}{W} = \text{diagonal}(\frac{1}{w_\ell}, \frac{1}{w_\ell}, \ldots, \frac{1}{w_0}, \frac{1}{w_0})$, depending on the order in which the response values are observed. The region R is the interval $[-1,1]$.

The $\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X}$ and $\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{y}$ matrices are,

$$\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{X} = \begin{bmatrix} a & 0 & b \\ 0 & b & 0 \\ b & 0 & c \end{bmatrix}, \quad \text{where} \quad \begin{aligned} a &= 2n_\ell w_\ell + n_0 w_0 \\ b &= 2n_\ell w_\ell \ell^2 \\ c &= 2n_\ell w_\ell \ell^4 \end{aligned}, \tag{3.2}$$

$$[\underset{\sim}{X}'\underset{\sim}{W}^{-1}\underset{\sim}{y}]' = [w_\ell(\underset{x=-\ell}{\Sigma y} + \underset{x=+\ell}{\Sigma y}) + w_0 \underset{x=0}{\Sigma y},\ \ell w_\ell(\underset{x=+\ell}{\Sigma y} - \underset{x=-\ell}{\Sigma y}),\ \ell^2 w_\ell(\underset{x=-\ell}{\Sigma y} + \underset{x=\ell}{\Sigma y})]'. \tag{3.3}$$

The notation $\underset{x=-\ell}{\Sigma y}$ in (3.3) denotes the sum of the observations $y_1, y_2, \ldots, y_{n_\ell}$ at $x = -\ell$. From (2.4), (2.6) and (2.7),

$$\Omega^{-1} = 2,\ \underset{\sim}{\mu}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix},\ \underset{\sim}{\mu}_{12} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \text{ and } \underset{\sim}{A} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.4}$$

From (2.8),

$$\text{Min } B = \frac{(2n_\ell + n_0)}{\sigma^2}\{\frac{4}{45}\}\beta_2^2, \tag{3.5}$$

which is independent not only of ℓ but also of w_ℓ and w_0. The expression (2.15) for V is,

$$V = (2n_\ell + n_0)\{\frac{6n_\ell w_\ell \ell^2(3\ell^2-2) + n_0 w_0(3\ell^2+1) + 2n_\ell w_\ell\}}{18 n_0 w_0 n_\ell w_\ell \ell^4} \tag{3.6}$$

The minimum value of V, using the method of minimum bias estimation, denoted by V(E) is,

$$V(E) = \frac{(2n_\ell+n_0)\ \{16n_\ell w_\ell - n_0 w_0\}}{8n_\ell w_\ell (2n_\ell w_\ell + n_0 w_0)} \quad , \tag{3.7}$$

and is attained when,

$$\ell^2(E) = \frac{2}{3}\ \frac{(2n_\ell w_\ell + n_0 w_0)}{4n_\ell w_\ell - n_0 w_0} \quad . \tag{3.8}$$

Now, if we fit a first degree model $\hat{y}(\underset{\sim}{x}) = \underset{\sim}{x}_1 \underset{\sim}{b}_1$, the weighted least squares estimator $\underset{\sim}{b}_1$ is obtained by,

$$\underset{\sim}{b}_1 = (\underset{\sim}{X}_1' \underset{\sim}{W}^{-1} \underset{\sim}{X}_1)^{-1} \underset{\sim}{X}_1' \underset{\sim}{W}^{-1} y \quad , \tag{3.9}$$

where,

$$\underset{\sim}{X}_1' \underset{\sim}{W}^{-1} \underset{\sim}{X}_1 = \begin{bmatrix} 2n_\ell w_\ell & 0 \\ 0 & 2n_\ell w_\ell \ell^2 \end{bmatrix} \quad . \tag{3.10}$$

To achieve minimum B, we have to set the design settings $x = \pm\ell$ so that condition (2.11) holds. The equality in (2.11) holds when,

$$\ell^2(D) = \frac{2n_\ell w_\ell + n_0 w_0}{6n_\ell w_\ell} \quad , \tag{3.11}$$

where $\ell^2(D)$ means the value of ℓ^2 met through the design. If we substitute the expression (3.11) into the following expression for V which is the average variance of $\hat{y}(\underset{\sim}{x})$ for a first degree model,

$$V = N \text{ trace } \{(\underset{\sim}{X}_1' \underset{\sim}{W}^{-1} \underset{\sim}{X}_1)^{-1} \underset{\sim}{\mu}_{11}\} \quad , \tag{3.12}$$

the resulting value of V, denoted by V(D), is,

$$V(D) = \frac{2(2n_\ell + n_0)}{2n_\ell w_\ell + n_0 w_0} \tag{3.13}$$

We can now compare the method of minimum bias estimation to the method of Box and Draper using V(E) and V(D) which are minimum at the respective optimum values of $x = \pm \ell$. The comparison can be made by substituting the expressions (3.7) and (3.13) for V(E) and V(D) respectively in the ratio,

$$\frac{V(E)}{V(D)} = \frac{16n_\ell w_\ell - n_0 w_0}{16n_\ell w_\ell} \tag{3.14}$$

which is obviously less than or equal to one. To evaluate the ratio (3.14) for different values of n_ℓ, w_ℓ, n_0 and w_0, we let $w_\ell = kw_0$, $w_0 = 1$ and vary the constants k, n_ℓ and n_0. In Table I, on the following page, some results of this study are presented. (It might be worth noting that certain values of $\ell(E)$ for small $k < 1$ lie outside the region R.)

Table I.

n_0	n_ℓ	k	V(E)	ℓ(E)	V(D)	ℓ(D)	V(E)/V(D)	@ ℓ(D) V(E)/V(D)
1	1	.3	3.000	2.31	3.75	0.94	0.80	1.000
		.5	2.615	1.15	3.00	0.82	0.875	0.996
		1	1.875	0.82	2.00	0.71	0.937	0.996
		2	1.163	0.69	1.20	0.65	0.969	0.993
	2	.3	4.072	1.02	4.55	0.78	0.89	1.000
		.5	3.125	0.82	3.33	0.71	0.937	0.996
		1	1.937	0.69	2.00	0.65	0.969	
		2	1.094	0.63	1.11	0.61	0.984	
	3	.3	4.653	0.85	5.00	0.72	0.931	
		.5	3.354	0.73	3.50	0.67	0.958	
		1	1.958	0.65	2.00	0.62	0.979	
		2	1.066	0.61	1.077	0.60	0.989	
2	1	.6	2.000	2.31	2.50	0.94	0.80	
		1	1.750	1.15	2.00	0.82	0.875	
		2	1.250	0.82	1.331	0.71	0.937	
	2	.3	3.00	2.31	3.75	0.94	0.80	
		.6	2.443	1.02	2.73	0.78	0.896	
		1	1.875	0.82	2.00	0.71	0.937	
		2	1.163	0.69	1.20	0.65	0.969	

4. Summary

We have shown how minimum bias estimation can be used in a weighted least squares analysis. Using a very simple example, we

have illustrated how this criterion can be superior to the method of design construction suggested by Box and Draper for minimizing the sum of the average squared bias and the average variance of $\hat{y}(\underset{\sim}{x})$ over a region of interest R.

In defense of Box and Draper's methodology however, Box and Draper fit a model of degree d_1 and concentrate on the possibility of underestimating the true surface which might be represented more adequately by a polynomial of degree $d_2 (d_2 > d_1)$. In using minimum bias estimation, on the other hand, one considers fitting a model of degree d_2 initially (in terms of deciding on the design coordinates) and then one uses an equation of d_1 to predict the response. To use minimum bias estimation, it is assumed one has knowledge (in terms of the degree of the true polynomial) about the shape of the true surface, an assumption not necessarily made by Box and Draper.

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A SURVEY OF PROCEDURES FOR TESTS OF SEPARATE FAMILIES OF HYPOTHESES

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Origin of Problem

The problem presented herein was undertaken while the author was on active duty with the U. S. Army at the U. S. Army Aberdeen Research and Development Center, Aberdeen Proving Ground, Maryland. The author's interest in the problem arose when one of his colleagues asked him how to determine which of two parametric families provides the best fit for a given set of data. The author, unable to answer this question and finding little help in the literature, set about to characterize possible procedures for handling the problem and then to compare them for several pairs of families using Monte Carlo samples as data. The principal intent of this paper is to provide a more or less practical guide for dealing with this type of problem.

The following example demonstrates more clearly the type of problem with which we are dealing.

When patients suffering from cancer are treated by surgery or radiotherapy, some of them respond favorably to treatment and may subsequently enjoy a period of months or years completely free from any signs or symptoms of the disease. In other cases, the tumor persists in spite of treatment, and in due course causes the death of the patient. In some cases although immediate response to treatment is good, the disease still recurrs at some later time. In assessing the value of methods of treatment, it is of primary importance to be able to estimate the proportion of patients permanently cured. A common method of presenting results has been to simply quote the five-year survival rate. But because recurrences of the disease can and do occur subsequent to this five year period, this estimate is somewhat crude.

It is important to be able to estimate the survival rate beyond the five year period, and in addition, it is important to be able to estimate the rate on the basis of data observed less than five years. Five years is too long to wait to be able to estimate the effectiveness of a new treatment. For this type of analysis one needs to know the distribution of the survival times for those patients not permanently cured.

Boag in 1949 proposed a method of maximum likelihood for estimating the proportion of patients cured by therapy. His method requires the choice of a particular distribution for the survival times. For his data Boag compared the lognormal and exponential distributions, and made his choice by comparing the p-values for two chi-square tests, i.e., he compared the test statistic for the chi-square test for lognormality with the test statistic for the chi-square test for exponentiality. His method of choosing between these two distributions, although having some natural appeal, is far from efficient.

Problem Description

Let $X_1, X_2, \ldots, X_n$ be a random sample from some unknown distribution F, and let it be desired to test the hypothesis

(1.1) $$H_1: \ F \in \mathcal{F}_1$$

where $\mathcal{F}_1$ is a family of probability distributions (such as the normal) with density f_1, against the hypothesis

(1.2) $$H_2: \ F \in \mathcal{F}_2$$

where $\mathcal{F}_2$ is another family of probability distributions having density f_2. For example, we might wish to choose between the normal and uniform distributions for our sample.

There are essentially two approaches that one can take to such a problem. One can treat it as a classification problem in which the two hypotheses are treated symmetrically and no null hypothesis is chosen, or one can choose a null hypothesis, fix a type I error, and treat it as a standard hypothesis testing problem. We will discuss the classification

aspect only here.

Our primary interest is in location-scale families where the densities have the form

$$\sigma^{-1} f_i((x-\mu)/\sigma), \quad i = 1, 2, \quad \mu \in R, \quad \sigma > 0. \tag{1.3}$$

The reason that we consider location-scale families is that for such families, the decision problem is invariant under transformations in location and scale. And in addition, there exists an optimal procedure among the class of invariant procedures for such problems.

Decision Theory

The action space $\mathcal{A}$ consists of the two actions

$$\begin{aligned} &a_1\text{--Choose } H_1 \\ &a_2\text{--Choose } H_2. \end{aligned} \tag{1.4}$$

For the loss function, we take simple loss, i.e., if $L(a_i,H_j)$ is defined to be the loss incurred from taking action a_i when hypothesis H_j is true, then we have

$$L(a_i,H_j) = \begin{cases} 1 & i \neq j \\ 0 & i = j \end{cases} \qquad i = 1, 2, \quad j = 1, 2. \tag{1.5}$$

We assume that each hypothesis is equally likely a priori to be true, so that for the prior distribution we take

$$\tau(H_1) = \tau(H_2) = 1/2. \tag{1.6}$$

Let α and β denote the type I and type II error probabilities, respectively, where for any procedure

(1.7) $$\alpha = P\{\text{Choose } H_2 \mid F \in \mathcal{F}_1\}$$

and

(1.8) $$\beta = P\{\text{Choose } H_1 \mid F \in \mathcal{F}_2\} .$$

For an invariant procedure α and β are independent of the values of any unknown parameters. And the Bayes risk, $r(\tau)$, under the foregoing theory, is

(1.9) $$r(\tau) = (\alpha+\beta)/2.$$

Thus, the Bayes risk is just the probability of misclassification, and the best procedure is that procedure which minimizes this probability. Thus, to compare procedures, one need only compare the respective Bayes risks.

Let $\varphi(\underset{\sim}{x})$ be a decision rule, where $\varphi(\underset{\sim}{x})$ is defined to be the probability of taking action a_1 after observing $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, i.e., if

(1.10) $$\varphi(\underset{\sim}{x}) = 1 \quad \text{take action } a_1$$

and if

(1.11) $$\varphi(\underset{\sim}{x}) = 0 \quad \text{take action } a_2.$$

Let Z be any test statistic for a classification procedure, then the principal decision rules considered in this study are

(1.12) $$\varphi(\underset{\sim}{x}) = \begin{cases} 1 & Z \geq 1 \\ 0 & Z < 1 \end{cases}$$

and

$$\varphi(\underset{\sim}{x}) = \begin{cases} 1 & Z \leq 1 \\ 0 & Z > 1 \end{cases} . \tag{1.13}$$

It is important to note that these decision rules do not depend on the sampling distributions of the test statistic. Thus, these rules can be used in the absence of any knowledge of sampling distributions.

Procedures

Best Invariant Procedure

The test statistic for the best invariant procedure when the densities are of the form (1.3) is

$$\delta(\underset{\sim}{x}) = \frac{\int_0^\infty \int_{-\infty}^\infty \sigma^{-n} \prod_{j=1}^n f_1((x-\mu)/\sigma) \, d\mu d\sigma/\sigma}{\int_0^\infty \int_{-\infty}^\infty \sigma^{-n} \prod_{j=1}^n f_2((x-\mu)/\sigma) \, d\mu d\sigma/\sigma} . \tag{1.14}$$

The decision rule for this procedure is given by(1.12) with $Z = \delta(\underset{\sim}{x})$. For the derivation of the test statistic for this procedure, see Zidek (1969). The best invariant procedure is the optimal procedure among the class of invariant procedures for the decision problem we have defined.

Likelihood Ratio

The test statistic for the likelihood ratio procedure is

$$\psi(\underset{\sim}{x}) = \frac{\max_{\mu,\sigma} \left\{ \sigma^{-n} \prod_{j=1}^n f_1((x-\mu)/\sigma) \right\}}{\max_{\mu,\sigma} \left\{ \sigma^{-n} \prod_{j=1}^n f_2((x-\mu)/\sigma) \right\}} . \tag{1.15}$$

The decision rule is of course that given by (1.12).

The next five procedures are all adaptations of goodness of fit tests. And they are to some extent procedures that one might naturally be led to consider.

Let

$$\epsilon(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{1.16}$$

and let

$$F_n(x) = n^{-1} \sum_{j=1}^{n} \epsilon(x - x_j) \tag{1.17}$$

be the sample empirical distribution function.

Let

$$\hat{F}_i(x) = F_i((x - \hat{\mu})/\hat{\sigma}), \qquad i = 1, 2, \tag{1.18}$$

be estimates of the distribution functions under the two hypotheses, where $\hat{\mu}$ and $\hat{\sigma}$ are estimates of the parameters. ($\hat{\mu}$ and $\hat{\sigma}$ might be the maximum likelihood estimates, or the minimum variance unbiased estimates, etc.)

Kolmogorov-Smirnov

Let

$$K_{ni} = \sup_{-\infty < x < \infty} \left| F_n(x) - \hat{F}_i(x) \right| \qquad i = 1, 2, \tag{1.19}$$

then K_{n1} is a Kolmogorov-Smirnov goodness of fit statistic for testing $H_0 = H_1$, and K_{n2} is a Kolmogorov-Smirnov goodness of fit statistic for testing $H_0 = H_2$.

A natural test statistic for the classification problem of testing H_1 against H_2 is just the ratio of K_{n1} and K_{n2}, since under H_1 we would expect K_{n1} to be less than K_{n2}, and under H_2 we would expect K_{n2} to be less than K_{n1}. Thus, as the test statistic for this procedure, we take

$$K_n = K_{n1}/K_{n2} \tag{1.20}$$

with the decision rule given by (1.13).

Cramer-Smirnov

Let

$$W_{ni}^2 = 1/12n + \sum_{j=1}^{n} (\hat{F}_i(x_{(j)}) - (2j-1)/2n) \qquad i = 1, 2. \tag{1.21}$$

In a manner analagous to the Kolmogorov-Smirnov procedure, we define

$$W_n^2 = W_{n1}^2/W_{n2}^2 \tag{1.22}$$

as the test statistic for the Cramer-Smirnov classification procedure. The decision rule is of course that given by (1.13). (See Darling (1957) for a discussion of the usual Cramer-Smirnov goodness of fit statistic.)

Anderson-Darling

For this procedure we have

$$A_{ni}^2 = -n - n^{-1} \sum_{j=1}^{n} (2j-1)\left\{\log(\hat{F}_i(x_{(j)})) + \log(1 - \hat{F}_i(x_{(n-j+1)}))\right\} \tag{1.23}$$

for $i = 1, 2$. As the classification test statistic we have

(1.24) $$A_n^2 = A_{n1}^2 / A_{n2}^2 .$$

This is an adaptation of the weighted Cramer-Von Mises test. See Anderson and Darling (1953), (1954).

Pearson-Durbin

This procedure is derived from Durbin's (Durbin (1961)) modification of Pearson's probability product test. Let

(1.25) $$c_{1i} = \hat{F}_i(x_{(1)}),\ c_{ji} = \hat{F}_i(x_{(j)}) - \hat{F}_i(x_{(j-1)}),\ j = 2,\ldots,n,$$

$$c_{n+1i} = 1 - \hat{F}_i(x_{(n)})$$

for i = 1, 2. The c's are the sample spacings under the two hypotheses.

Define

(1.26) $$w_{ji} = \sum_{r=1}^{j-1} c_{(r)i} + (n+j-2)\, c_{(j)i} ,\ j=1,\ldots,n,\ i=1,2,$$

and

(1.27) $$P_{mi} = -2 \log \prod_{r=1}^{n} w_{ri} , \qquad i = 1, 2.$$

The test statistic for this procedure is then just

(1.28) $$P_m = P_{m1} / P_{m2} ,$$

where the decision rule is again given by (1.13).

If we were dealing with completely specified hypotheses, then the w's for each distribution would have the same distribution as order statistics from a uniform distribution.

Modified Pearson-Durbin

This is a modification of the previous procedure for which we now choose

$$c_{n+1i} = 0 \qquad i = 1, 2, \tag{1.29}$$

instead of the values given in (1.25).

We then have

$$w'_{ji} = \sum_{r=2}^{j-1} c'_{(r)i} + (n+2-j)\, c'_{(j)i} \qquad j=2,\ldots,n,\ i=1,2, \tag{1.30}$$

and

$$MP_{mi} = -2 \log \prod_{r=2}^{n} w'_{ri} \qquad i = 1, 2. \tag{1.31}$$

(The index j runs from 2 here, since now $w_{1i} = 0$, for i = 1, 2.)

The test statistic is just

$$MP_m = MP_{m1}/MP_{m2} . \tag{1.32}$$

Binomial Interval

Let

$$\hat{f}_i(x) = \hat{\sigma}^{-1} f_i((x-\hat{\mu})/\hat{\sigma}), \qquad i = 1, 2, \tag{1.33}$$

be estimates of the density functions under each hypothesis.

Let

$$\hat{L}(x) = \hat{f}_1(x)/\hat{f}_2(x) , \tag{1.34}$$

$$I = \left\{x:\ \hat{L}(x) \geq 1\right\} , \tag{1.35}$$

(1.36) $$p_i = \int_I \hat{f}_i(x)\, dx \qquad i = 1, 2,$$

(1.37) $$\omega(t) = \begin{cases} 1 & \text{if } t \in I \\ 0 & \text{if } t \notin I \end{cases}$$

and

(1.38) $$m = \sum_{j=1}^{n} \omega(x_j) .$$

As the test statistic for this procedure, we then take

(1.39) $$BLR_1 = p_1^m(1-p_1)^{n-m}/p_2^m(1-p_2)^{n-m} .$$

The decision rule for this procedure is (1.12).

What this procedure does is essentially reduce H_1 and H_2 to the two simple hypotheses

(1.40) $$H_1: \quad p = p_1$$

and

(1.41) $$H_2: \quad p = p_2$$

where p is the true probability associated with the interval I.

All of these procedures are invariant under transformations in location and scale.

Monte Carlo Comparisons

To compare the relative performance of these procedures we generated Monte Carlo samples from several pairs of families for several sample sizes. We then used these Monte Carlo samples to estimate the Bayes risks for

each procedure. In general we generated 500 samples from each hypothesized family for each of four or five sample sizes.

Four or five sample sizes were considered for each family, because in addition to the results for each sample size for comparing procedures, we desired a measure of overall effectiveness. To obtain this measure, we used weighted regression techniques to fit the equation

$$(1.42) \qquad -\log(\alpha+\beta) = Cn$$

to the points

$$(1.43) \qquad (-\log(\hat{\alpha}_i+\hat{\beta}_i), n_i) \qquad i = 1, \ldots, k$$

for each procedure, where k is the number of sample sizes and $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the the estimates of α and β, respectively, for the ith sample size. This fitted equation is then used to give the approximate sample size necessary for any Bayes risk for each procedure. If $\hat{C}_1$ and $\hat{C}_2$ are the estimates of C for any two procedures, then the estimated relative efficiency of the second procedure with respect to the first is just

$$(1.44) \qquad \hat{E} = \hat{C}_2/\hat{C}_1 \ .$$

It should be noted that this measure of efficiency does not depend on the particular value of the Bayes risk.

We do not claim that equation (1.42) represents the true relation between sample size and Bayes risk for these procedures. However, it does seem to fit the data reasonably well, and thus does provide a proper means for making an overall comparison of these procedures. In all cases we did examine the residual sum of squares from the fit. In most cases this

number indicated a reasonable fit. However, as expected, in some cases the fit was not very good.

Results

Four pairs of families considered in this study are

(1.45)

Normal-Uniform--invariant
Normal-Exponential--invariant
Normal-Double Exponential--invariant
Lognormal-Exponential--not invariant .

Of these four pairs of families, only the lognormal-exponential pair does not yield an invariant decision problem. We consider this last pair, because both of these distributions are common in reliability theory, and because one will not always hypothesize families which are location and scale invariant.

Table 1 presents the relative efficiencies of these procedures for the four pairs of families listed above. In three of the cases the relative efficiencies are with respect to the best invariant procedure, and in the fourth case, the efficiencies are with respect to the likelihood ratio procedure.

If we examine the results of Table 1, a result to note is the poor performance of the likelihood ratio procedure for the case of the normal and uniform hypotheses. This procedure does very poorly for this case, because the estimates of the parameters for the uniform distribution are systematically biased. This same type of problem also makes the results for the normal-exponential case for the likelihood ratio procedure somewhat inefficient.

TABLE 1

RELATIVE EFFICIENCIES

Procedure	N-U	N-E	N-D	L-E
1) Best Invariant	1.000	1.000	1.000	---
2) Likelihood Ratio	.348	.834	.987	1.000
3) Pearson-Durbin	.971	.941	.785	.791
4) Modified Pearson-Durbin	.976	.935	.932	.868
5) Binomial Interval	.760	.724	.898	.693
6) Anderson-Darling	.384	.783	.906	.694
7) Cramer-Smirnov	.319	.694	.838	.707
8) Kolmogorov-Smirnov	.266	.642	---	.628

Computational Complexity of Procedures

We are also interested in comparing the procedures on the basis of the computational ease in the determination of the test statistic. If we have two procedures with the same efficiency, then we would prefer the procedure which is the simpler to compute. The following list gives the hierarchy of difficulty of the computations involved in these procedures, with the list being in order of increasing difficulty.

(1.46)

I) Best Invariant
Likelihood Ratio
Binomial Interval

II) Kolmogorov-Smirnov
Cramer-Smirnov

III) Pearson-Durbin
Modified Pearson-Durbin
Anderson-Darling

The easiest test statistic to calculate is that for the best

invariant procedure. One should, however, work with the logarithm of this test statistic. Both the likelihood ratio and binomial interval test statistics are quite easy to obtain. The characteristic that differentiates groups II and III from group I is the necessity of obtaining the probability integrals at each of the order statistics. The characteristic which separates group II from group III is the need for the calculation of a large number of logs for the procedures in group III.

Procedures Recommended for Use

1) Best Invariant--This is the best procedure for invariant decision problems and should be used whenever the decision problem is invariant.
2) Likelihood Ratio--This procedure should be used for non-invariant decision problems if F_1 and F_2 are absolutely continuous with respect to one another. (Avoid this procedure in cases like the normal-uniform.)
3) Modified Pearson-Durbin--This procedure should be used for non-invariant decision problems when the families hypothesized are not absolutely continuous with respect to each other.
4) Binomial Interval--This procedure should be used in place of the Modified Pearson-Durbin procedure when the values of the probability integrals at each of the order statistics are difficult to obtain.

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MAXIMUM LIKELIHOOD ESTIMATION FROM RENEWAL TESTING

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ABSTRACT

This paper considers the maximum likelihood estimates of life-time distributions over an interval [0,T) from the following time truncated experiment. At time zero, the beginning of the testing n ($1 \leq n < \infty$) items are put on test. When an item fails it is replaced and at time T all testing is stopped.

Assumptions about the form of the life-time distribution on [0,T) are required. Distributions considered are:

(1) A single parameter class which includes the Weibull family;

(2) A multiple parameter class with increasing failure rate on [0,T);

(3) A nonparametric class which includes the increasing failure rate family.

Useful and desirable properties of the maximum likelihood estimates are shown.

1. INTRODUCTION AND SUMMARY

Based on a number of practical reasons it is often necessary and even desirable in life testing (reliability) studies to fix the total testing time, say, at T ($T < \infty$), before testing begins. For example, an experimenter would rarely use a testing plan that did not limit the total testing time when the items being tested can be assumed very reliable, since the testing time would usually be very long. The total testing must, also, be limited if project deadlines must be met, or if equipment or personnel used in the testing can only be spared for some specified length of time.

Limiting the total testing time need not, however, be contrary to the goals of the experimenter. For example, if the experimenter can assume that the general form of the life-time distribution belongs to some parametric class defined on the nonnegative real axis, then limiting the testing time to T,

he can still estimate the unknown parameters of the distribution on $[0, \infty)$.

If the experimenter cannot assume that the life-time distribution has a particular form on $[0, \infty)$ but only on $[0, T)$, then he must limit his inferences to the latter interval. However, if $[0, T)$ includes the mission time of the items tested then, for all practical purposes, he need not infer anything about the distribution outside this interval.

One of the most popular time truncated testing plans is the subject of the present paper. This plan stimulates that n items are initially put on test at time zero. When an item fails it is replaced by a new item and at time T all testing is stopped. (For convenience this plan is called "Testing Plan A".) Renewing a failed item is a method to further save experimental time and generally results in a better utilization of equipment and personnel.

Practically all of the statistical procedures developed in the literature for Testing Plan A are based on the assumption that the underlying life-time distribution of the items tested is the exponential law

$$G(x) = 1 - \exp(-\lambda x), \tag{1.1}$$

$\lambda > 0$, $x \geq 0$. (See Epstein [1959] for a review of these procedures.) In practice, however, the exponential assumption is often not valid since it implies a no wear-out (or no aging) property of the items. Moreover, if the times to failure of the items do not follow the law (1.1) these exponential procedures could possibly be sensitive to this departure. (See, for example, Zelen and Danmiller [1961]).

For Testing Plan A the present paper investigates the maximum likelihood estimates of life-time distributions from three general classes. Life-time distributions describing wear-out are contained in each of these classes and, also, each class contains the exponential distribution.

Specifically, in Section 3 the maximum likelihood estimate (MLE) of the parameter λ will be considered when the life-time distribution has the form

$$F(x) = 1 - \exp(-\lambda g(x)),$$

$\lambda > 0$, $0 \leq x < T$, $g(\cdot)$ is a known, strictly increasing, differentiable function on $[0, T)$ with $g(0) = 0$. Observe that nothing is assumed about F on $[T, \infty)$. This parametric class is obviously relevant to life testing since, for example, it includes the exponential (when $g(x) = x$, $x > 0$), the Weibull (when $g(x) = x^{\beta}$, $\beta > 0$, $x > 0$) and the extreme-value distributions (when $g(x) = e^{x} - 1$, $x > 0$). Asymptotic distribution theory, which will allow one to test hypothesis on the true value of λ, shall be given along with a number of pleasant properties of the MLE. These results do not depend on the fact that F is not restricted on $[T, \infty)$. Also, a major drawback to another method of estimating λ shall be discussed.

As such, the class of distributions introduced in Section 4 has not been considered in the literature. Practical applications of this class shall be discussed and the MLE's of parameters determining the life-time distributions over $[0, T)$ are shown to be asymptocially normal and consistent.

Often an experimenter does not know a priori that the law governing the times to failure of the items tested belongs to a certain parametric class. He may, however, know that the underlying distribution is a member of a non-parametric class of distribution, e.g., the increasing (decreasing) failure rate (IFR (DFR)) family.

Marshall and Proschan (1965) considered the MLE of a life-time distribution, assuming only that it was a member of the IFR (DFR) family and that data arise from a testing plan which does not allow censoring, time-truncation or replacement. Bray, Crawford and Proschan (1967), also, considered the MLE of a life-time distribution from a nonparametric class which includes both the IFR and DFR families. The testing plan they introduced allowed for the consideration of various types of incomplete data.

Since nonparametric estimation has not been considered in the literature for Testing Plan A, we will study this type of estimation in Section 5. The class of distributions considered includes the IFR family and the main result of that section is the consistency of the MLE over $[0,T)$.

2. PRELIMINARIES

In this section preliminary definitions and notations needed in later sections shall be collected. For completeness we give

Definition 2.1 (Testing Plan A)

At time zero, the beginning of the testing, n new items from a population are put on test. When an item fails it is instantaneously replaced with a new item from the original population and at time T the testing is stopped.

Estimation from Testing Plan A has been considered by several authors, including Epstein (1959, page 3.17) and Gnedenko, Belyayev and Solovyev (1969, page 169), when the life-time distribution of the items is exponential. These authors derived the likelihood function by standard methods which could, also, be used for other classes of distributions with densities. In the present approach, however, the derivation of the likelihood function utilizes the theory of stopping variables. The benefits of this approach are two-fold. Firstly, a straightforward method of obtaining the likelihood function is developed for the parametric classes of distributions considered in Sections 3 and 4. Finally, this approach motivates a generalized likelihood function needed in Section 5 for the nonparametric class.

To develop this preliminary theory observe that Testing Plan A may be considered as n independent experiments, each beginning at time zero and ending at time T. Throughout this paper K_r will denote the random number of items put on test in the r-th experiment and X_{ir} will denote the time to failure of the i-th item put on test in this experiment, $i = 1, 2, \ldots,$ $r = 1, \ldots, n$. From this notation we have that K_r is the first integer such that

$$\sum_{i=1}^{K_r} X_{ir} \geq T,$$

$r=1, \ldots, n$.

For the moment consider only the 1-st experiment and let $X_i = X_{i1}$, $i=1, 2, \ldots, K=K_1$. Also, let F be the cumulative distribution

r=1, ..., n. Since the testing is truncated at time T, experiment 1 is characterized by the time on test statistics $(Y_1, \ldots, Y_k)$. By (2.2) and (2.3) one sees that the experiment is equivalently characterized by the times to failure $(X_1, \ldots, X_{K-1})$.

Now, in almost all cases where the c.d.f. has a probability density function (p.d.f) one can show that the likelihood function is derived from the integrand of an expression equated to 1 and where the integration is over the sample space of the random variables of interest. For $(X_1, \ldots, X_K)$ the sample space is

$$\Omega_1 = \bigcup_{k=1}^{\infty} A_k \, B_k,$$

and for $(X_1, \ldots, X_{K-1})$ the sample space is

$$\Omega_2 = \bigcup_{k=1}^{\infty} A_k.$$

We, therefore, integrate out x_k in (2.1) obtaining

$$1 = \sum_{k=1}^{\infty} \int_{A_k} \cdots \int [1-F(\{T - \sum_{i=1}^{k-1} x_i\}^-)] \prod_{j=1}^{k-1} dF(x_i), \tag{2.4}$$

where

$$F(x^-) = \lim_{\varepsilon \longrightarrow 0} F(x-\varepsilon), \ \varepsilon > 0,$$

and it is observed that

$$B_k = \{x_1, \ldots, x_k: \ x_k \geq T - \sum_{i=1}^{k-1} x_i\},$$

$k = 1, 2, \ldots$.

Hence, if F is absolutely continuous on [0, T) with p.d.f. f, then

$$1 = \sum_{k=1}^{\infty} \int \cdots \int_{A_k} [1-F(\{T - \sum_{i=1}^{k-1} x_i\}^-)] \prod_{j=1}^{k-1} f(x_j)\, dx_j. \qquad (2.5)$$

This motivates the following:

Definition 2.2.

If the times to failure of the items are independent and identically distributed (iid) with c.d.f. F, F(0) = 0, and F is absolutely continuous on [0, T) with p.d.f. f, then the likelihood function L for Testing Plan A is

$$L = \prod_{r=1}^{n} L_r, \qquad (2.6)$$

where

$$L_r \equiv L_r(X_{ir}, \ldots, X_{K_r-1,\, r}),$$

is the random variable,

$$L_r = [1-F(\{T - \sum_{i=1}^{K_r-1} X_{ir}\}^-)] \prod_{j=1}^{K_r-1} f(X_{jr}). \qquad (2.7)$$

Equation (2.6) is a result of the independence of the n experiments and equation (2.7) is obtained from the integrand of equation (2.5) when the random variables replace their corresponding sample points. Definition 2.2 will be used in Section 3 and 4 to derive the MLE's for the parametric classes of distributions. A generalized definition of MLE, based on equation (2.4), will be defined in Section 5 for the nonparametric class.

3. SINGLE PARAMETER ESTIMATION

Introduction

Throughout this section it will be assumed that the underlying c.d.f. of the times to failure is

$$F(x) = 1 - \exp(-\lambda g(x)) \tag{3.1}$$

for $0 \leq x < T$, $\lambda > 0$, $g(\cdot)$ is known and strictly increasing with $g(0) = 0$ and derivative $g'(x)$ $0 < x < T$. We shall derive the MLE $\hat{\lambda}_n$ of λ and show that: (a) $\hat{\lambda}_n$ is strongly consistent, as $n \to \infty$; (b) $\hat{\lambda}_n$ is asymptotically normally distributed as $n \to \infty$; (c) $\hat{\lambda}_n$ is asymptotically efficient, as $n \to \infty$. Also, a major drawback to some previously published work dealing with the estimation of λ shall be discussed.

Previous Work

Gnedenko, Belyayev and Solovyev (1969), devoted an entire section of their book, beginning on page 168, to the MLE of the parameter λ when the underlying distribution is the exponential law

$$G(x) = 1 - \exp(-\lambda x) \tag{3.2}$$

$x > 0$, $\lambda > 0$, for six life testing plans, one of which was Testing Plan A. Observe, now, that if X is a random variable with c.d.f. given by (3.1) for $x > 0$ then $g(X)$ is a random variable with c.d.f. given by (3.2). Noting this, Gnedenko, et al mentioned that if the life-time distribution is given by (3.1) for $x > 0$ then one may make the transformation $Y = g(X)$ on the data and use their exponential procedures to estimate λ. However, it was not pointed out that if the exponential procedures are used for Testing Plan A then the total testing time will not necessarily be T, which violates the purpose of this testing plan.

To see this difficulty, observe that the suggested test plan implies that one make the transformation

$$W_{ir} = g(X_{ir})$$

$i \geq 1$, r=1, ..., n, choose a constant $C > 0$, and continue testing in the r-th experiment (r=1, ..., n) until time C on the $g(\cdot)$ time axis. If one does this then the random number of items, K_r, put on test in the r-th experiment is the first integer such that

$$\sum_{i=1}^{K_r} W_{ir} \geq C,$$

r=1, ..., n. Therefore, the actual (untransformed) total testing time in the r-th experiment is

$$\sum_{i=0}^{K_r-1} X_{ir} + g^{-1}\left(C - \sum_{i=0}^{K_r-1} g(X_{ir})\right),$$

r=1, ..., n. If C is to be chosen such that the total testing time is T, then for $K_r = 1$, the total testing time is $g^{-1}(C) = T$. Thus, $g(T)$ is the only candidate for C. Now, if

$$\sum_{i=0}^{K_r-1} X_{ir} + g^{-1}\left(g(T) - \sum_{i=0}^{K_r-1} g(X_{ir})\right) = T$$

for $K_r > 1$, this would imply that

$$g(T) - \sum_{i=0}^{K_r-1} g(X_{ir}) = g\left(T - \sum_{i=0}^{K_r-1} X_{ir}\right),$$

which is, in general, not true for non-linear g. Thus, when one makes such a transformation the total testing times for the n experiments will generally be random variables. This violates the purpose of Testing Plan A which is to fix the total testing time at T. The work presented in this section allows one to estimate λ without using such a transformation and, hence, avoiding this difficulty.

The MLE of λ

The MLE $\hat{\lambda}_n$, say, of λ shall now be derived. In what follows let

$$g(T) = \lim_{\varepsilon \to 0} g(T-\varepsilon), \quad \varepsilon > 0.$$

Lemma 3.1

The MLE $\hat{\lambda}_n$ of λ is

$$\hat{\lambda}_n = \sum_{r=1}^{n} (K_r - 1) / \sum_{r=1}^{n} \sum_{r=1}^{K_r} g(Y_{ir}). \tag{3.3}$$

proof

By Definition 2.2 the likelihood function is

$$L = \prod_{r=1}^{n} L_r$$

where

$$L_r = \prod_{i=1}^{K_r - 1} \lambda g'(Y_{ir}) \exp(-\lambda g(Y_{ir}) \exp(-\lambda g(Y_{K_r r})),$$

r=1, ..., n. Maximizing L with respect to λ yields $\hat{\lambda}_n$ given by (3.3).

The reader should note that if F is continuous at T, then

$$\sum_{r=1}^{n} (K_r - 1)$$

is the number of failures in the n experiments. Also, if the times to failure are exponentially distributed (i.e., $g(x) = x$, $x > 0$), then

$$\sum_{r=1}^{n} \sum_{i=1}^{K_r} g(Y_{ir}) = nT.$$

Hence, $\hat{\lambda}_n$ is the usual estimator in the exponential case.

Observe, now, that identity (2.4) implies that the probability of <u>any event</u> associated with the outcome of an experiment based on Testing Plan A only depends on $F(x)$ for $0 \leq x < T$. Since these probabilities do not depend on $F(x)$ for $T \leq x < \infty$, it follows that the statistical properties of any random variable obtained from Testing Plan A are independent of $F(x)$ for $T \leq x < \infty$. We, therefore, have

<u>Theorem 3.2</u>

The statistical properties of λ_n and all other random variables obtained from Testing Plan A do not depend on the values of F on $[T, \infty)$.

Strong Consistency of λ_n

We will now show that $\hat{\lambda}_n$ converges to λ almost surely*(a.s.) as $n \to \infty$. To show this we will need the following results.

<u>Lemma 3.3</u>

If F_1 is any c.d.f. such that $F(x) = F_1(x)$, $0 \leq x < T$, then

$$P[K_1 = k|F] = P[K_1 = k|F_1], \text{ for all } k = 1, 2, \ldots$$

<u>Proof</u>:

The proof follows from Theorem 3.2.

---

*The term "almost surely" means that a certain event holds with probability one.

The following result is needed to show consistency and is, also, useful throughout the remainder of this section.

Theorem 3.4

$$\frac{1}{\lambda} E(K_1 - 1) = E\left(\sum_{i=1}^{K_1} g(Y_{i1})\right). \tag{3.4}$$

Proof:

Let $X_i = X_{i1}$, $i = 1,2,\ldots$, $K = K_1$. By Lemma 3.3 $E(K)$ does not depend on $F(x)$ for $x \geq T$. Hence, if $g(T) < \infty$ then we may extend $g(x)$ for $x \geq T$ in any manner we wish to keep F a c.d.f. and $E(K)$ will remain unchanged. We therefore assume that $g(x) = g(T) + (x-T)$ for $x \geq T$, when $g(T) < \infty$. Hence, whether or not $F(T) < 1$ or $F(T) = 1$, $g(X_i)$ has an exponential distribution with mean $1/\lambda$. By Wald's Lemma (1944)

$$\frac{1}{\lambda} E(K) = E\left(\sum_{i=1}^{K} g(X_i)\right). \tag{3.5}$$

$$\text{Now, } E(g(X_K)) = E\left(E\left(g(X_K) \Big| \sum_{i=1}^{K-1} X_i\right)\right)$$

$$= E\left(E\left(g(X) \mid X \geq T - \sum_{i=1}^{K-1} X_i\right)\right)$$

$$= E\left(E\left(g(X) \mid g(X) \geq g\left(T - \sum_{i=1}^{K-1} X_i\right)\right)\right)$$

where $g(X)$ is a random variable with c.d.f. $1-\exp(-\lambda y)$, $y \geq 0$. Thus,

$$E(g(X_K)) = E(\frac{1}{\lambda} + g(T - \sum_{i=1}^{K-1} X_i))$$

$$= \frac{1}{\lambda} + E(g(T - \sum_{i=1}^{K-1} X_i)). \qquad (3.6)$$

Equations (3.5) and (3.6) imply (3.4).

We may now prove

Theorem 3.5

The MLE, $\hat{\lambda}_n$, given by equation (3.3), is a strongly consistent estimator of λ as $n \to \infty$.

Proof:

By the strong law of large numbers, as $n \to \infty$

$$\sum_{r=1}^{n} (K_r - 1)/n \longrightarrow E(K_1 - 1) \text{ a.s.},$$

and

$$\sum_{r=1}^{n} \sum_{i=1}^{K_r} g(Y_{ir})/n \longrightarrow E(\sum_{i=1}^{K_1} g(Y_{i1})) \text{ a.s. .}$$

Hence, as $n \to \infty$

$$\hat{\lambda}_n = \frac{\sum_{r=1}^{n} \sum_{i=1}^{K_r} (K_r - 1)/n}{\sum_{r=1}^{n} \sum_{i=1}^{K_r} g(Y_{ir})/n} \longrightarrow \frac{E(K_1 - 1)}{E(\sum_{i=1}^{K_1} g(Y_{i1}))} \text{ a.s. .}$$

The result follows from equation (3.4).

Asymptotic Normality of λ_n

We will now show the asymptotically normality of the MLE $\hat{\lambda}_n$ for two different, but asymptotically equivalent, normalizing sequences. We begin with

Theorem 3.6

The asymptotic distribution of $(\hat{\lambda}_n - \lambda)/\sqrt{D/n}$ is Normal (0,1), as $n \to \infty$ where

$$D = \frac{\mathrm{Var}((K_1 - 1) - \lambda \sum_{j=1}^{K_1} g(Y_{j1}))}{E^2(\sum_{j=1}^{K_1} g(Y_{j1}))} \quad . \tag{3.7}$$

Proof:

Let $M_r = \sum_{j=1}^{K_r} g(Y_{jr})$, $V_r = (K_r - 1)$, $M(n) = \sum_{r=1}^{n} \frac{M_r}{n}$,

$V(n) = \sum_{r=1}^{n} \frac{V_r}{n}$, and let $\underset{\sim}{Z}_r$ be the two-dimensional random vector, $\underset{\sim}{Z}_r = (M_r, V_r)$, $r = 1, \ldots, n$. Also, let $H(a,b)$ be the function of the two variables a, b, $H(a,b) = a/b$. Now, $(M(n), V(n))$ is the first moment vector corresponding to the sample $\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_n$. By Cramér (1946, pages 353, 367), $H(V(n), M(n)) = \hat{\lambda}_n$ is asymptotically normal with asymptotic mean $\frac{E(V(n))}{E(M(n))} = \lambda$ by equation (3.4), and asymptotic variance

$$\frac{\mathrm{Var}(V(n))}{E^2(M(n))} - 2\,\mathrm{Cov}(V(n), M(n))\,\frac{E(V(n))}{E^3(M(n))} + \mathrm{Var}(M(n))\frac{E^2(V(n))}{E^4(M(n))} \quad .$$

Using equation (3.4) again the asymptotic variance equals

$$\frac{1}{nE^2\left(\sum_{j=1}^{K_1} g(Y_{j1})\right)} \left\{\mathrm{Var}(K_1-1) - 2\lambda \mathrm{Cov}\left(K_1-1, \sum_{j=1}^{K_1} g(Y_{j1})\right) + \lambda^2 \mathrm{Var}\left(\sum_{j=1}^{K_1} g(Y_{j1})\right)\right\}.$$

This completes the proof.

The next theorem will be useful in what follows.

Theorem 3.7

$$E(K_1-1) = \mathrm{Var}\left(K_1-1-\lambda \sum_{j=1}^{K_1} g(Y_{j1})\right) \tag{3.8}$$

Proof:

Let $K = K_1$, $X_i = X_{i1}$, $i = 1,\ldots, K-1$, and $Y_i = Y_{i1}$, $i = 1,\ldots, K$. Also, let

$$f(x) = \lambda g'(x) \exp(-\lambda g(x)),$$

and

$$p(x_1,\ldots,x_{k-1}|\lambda) = \prod_{j=1}^{k-1} f(x_j)\left[1-F\left(\left\{T-\sum_{i=1}^{k-1} x_i\right\}^-\right)\right]$$

It is easy to verify that

$$E\left(\frac{d}{d\lambda} \log p(X_1,\ldots,X_{K-1}|\lambda)\right)^2 = -E\left(\frac{d^2}{d\lambda^2} \log p(X_1,\ldots,X_{K-1}|\lambda)\right).$$

The left-hand side of (3.9) is equal to

$$E(\frac{K-1}{\lambda} - \sum_{j=1}^{K} g(Y_j))^2.$$

But using equation (3.4) we have

$$\mathrm{Var}(\frac{K-1}{\lambda} - \sum_{j=1}^{K} g(Y_j)) = E(\frac{K-1}{\lambda} - \sum_{j=1}^{K} g(Y_j))^2. \tag{3.10}$$

The right-hand side of (3.9) is equal to $E(K-1)/\lambda^2$. Hence, (3.8) follows.

Using equations (3.4) and (3.8) it follows, also, that

$$D = \lambda^2/E(K_1 - 1). \tag{3.11}$$

From this we have

Corollary 3.8

The asymptotic distribution of $(\hat{\lambda}_n-\lambda)/\sqrt{\lambda^2/(nE(K_1-1))}$ is Normal (0,1) as $n\to\infty$.

By the strong law of large numbers Corollary 3.8 gives

Corollary 3.9

The asymptotic distribution of $(\hat{\lambda}_n-\lambda)/\sqrt{\lambda^2/(\sum_{r=1}^{n}(K_r-1)}$ is Normal (0,1) as $n\to\infty$.

Asymptotic Efficiency of λ_n

Let $h(X_1,\ldots,X_{K-1})$ be an estimate of λ, where $X_i = X_{il}$, $i=1,\ldots, K-1$, $K = K_1$. Then

Theorem 3.10

$$D(1 + \frac{d}{d\lambda} B(\lambda))^2 \leq \mathrm{Var}(h(X_1,\ldots,X_{K-1})),$$

where $B(\lambda) = E(h(X_1,\ldots,X_{K-1})|\lambda) - \lambda$, and D is given by equation (3.7).

Proof:

It is straightforward to show that

$$[1 + \frac{d}{d\lambda} B(\lambda)]^2 \leq \mathrm{Var}(h(X_1,\ldots,X_{K-1})|\lambda)\ E(\frac{d}{d\lambda} \log p(X_1,\ldots,X_{K-1}|\lambda))^2.$$

[Note that in the sequential form of the Cramér-Rao bound that $\prod_{i=1}^{k} f(x_i)$ corresponds to $p(x_1,\ldots,x_{k-1}|\lambda)$].
Using equations (3.8), (3.10), and (3.11) yields the result.

This implies that if h_1 is an unbiased estimator of λ based on the outcomes of n experiments, then

$$\frac{D}{n} \leq \mathrm{Var}(h_1). \tag{3.12}$$

From this we have the following

Theorem 3.11

$\hat{\lambda}_n$ is an asymptotically efficient estimator of λ.

Proof:

Our concept of efficiency is the same as the concept given by BAN estimators for fixed sample size. (See Rao (1968), p. 284). The result then follows from Theorem 3.6 and inequality (3.12).

Comments

Note that if the times to failure of the items put on test actually have the c.d.f. $F(x) = 1 - \exp(-\lambda g(x))$ for $0 \leq x < T + b$, $0 < b \leq \infty$, then, of course, the assumptions required for $F(\cdot)$ are satisfied. In this case the estimate $\hat{\lambda}_n$ allows one to estimate $F(x)$ for $0 \leq x < T + b$ from data restricted to $[0,T]$. Suppose, however, that the c.d.f. has the form $F(x) = 1 - \exp(-\lambda g(x))$, $0 \leq a \leq x < T + a$. Then items of age a have the c.d.f. $G(x) = 1 - \exp(-\lambda h(x))$ $a \leq x < T + a$, where $h(x) = g(x) - g(a)$. Thus one may put items of age a on test at time 0 and use the theory presented in this section to estimate $F(x)$ for $a \leq x < T + a$.

It is to be remarked, also, that numerous computer simulation runs substantiate the conjecture that the MLE of λ is generally not unbiased; i.e., in general $E(\hat{\lambda}_n) \neq \lambda$. However, the bias approaches zero as n or T gets large.

4. MULTIPLE PARAMETER ESTIMATION

Introduction

In this section it is assumed that the c.d.f. F governing the times to failure of the items put on test is absolutely continuous on [0,T) with p.d.f. f and F(0) = 0. Also, it is assumed that the failure rate $f(x)/[1-F(x)] = \lambda_q$, for $x \in [S_q, S_{q+1})$, $q = 0,1,\ldots, t-1$, where $0 = S_0 < S_1 < \ldots < S_t = T$, and $0 \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{t-1} < \infty$. Thus, assuming that the S_q, $q = 0,1,\ldots,t$ are known, and data are collected from Testing Plan A, the MLE's of λ_{nq}, $q = 0, 1, \ldots, t-1$, are determined and shown to be strongly consistent estimators of λ_q as $n \to \infty$. The asymptotic normality of these estimators, as $n \to \infty$ is, also, established.

Consider now a situation when Testing Plan A and this class of distributions may be applicable. The guidance system or some other system or component in a rocket may have a failure rate which is constant when the booster of the first stage of the rocket is in operation. However, when the first stage falls away and the second stage booster is fired the failure rate of the system may change and in fact increase instantly to a constant value during this stage. If this is true for all stages of the rocket, then, (since the exact length of each stage and the exact time of the staging is known), laboratory

testing may be used to estimate the failure rate of the system for the duration of its mission.

Another possible application may arise when one is interested in estimating the failure rate of an electronic apparatus as a function of the amount of voltage. It is not unusual for electronic tubes and the like to have a constant failure rate when the voltage is constant. If the failure rate is a nondecreasing function of the voltage then one may estimate the failure rate for specific values of the voltage in the following way. Let the testing time T be fixed and let $v_0 < v_1 < \ldots < v_{t-1}$ be voltages which are of interest to the experimenter. Let λ_i, $i = 0, \ldots, t-1$ be the failure rate of the items when they are receiving voltage v_i. Also, let $[S_i, S_{i+1})$, $i = 0, \ldots, t-1$ be a partition of $[0,T)$. When an item is put on test it receives voltage v_0. If it operates without failure for time S_1 then the voltage is increased instantly to v_1. Similarly, if the item operates for time S_i, $i < t$, then the voltage is increased to v_i. When an item fails it is replaced instantly by another new item and the voltage is reduced to v_0. If this item operates for time S_1 without failure then the voltage is increased to v_1, and so on. This process is continued until time T. The theory presented in this section will allow one to estimate the λ_i, $i = 0, \ldots, t-1$.

Applications of this model may also be possible in the fields of drug testing and toxicology. For example, suppose one

is interested in the effect of a toxic agent such as DDT or the effect of radiation, which decompose at a very slow rate. The failure rate depends on the dosage level and may be taken as constant for reasonably short periods of time and nondecreasing as the dosage level increases. The dosage is sequentially increased at the end of these successive periods and the model presented in this section may be used to estimate the failure rates corresponding to the different dosage levels for the time period of interest.

The Naive MLE of the λ_q

We begin this section by finding the values of λ_q, $q = 0, \ldots, t-1$, which will maximize L, given by Definition 2.2, without the restriction that $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{t-1}$.

From Definition 2.2 the likelihood L_r for the r-th experiment is

$$L_r = [1-F(\{T - \sum_{i=1}^{K_r-1} X_{ir}\}^-)] \sum_{i=1}^{K_r-1} f(X_{ir}), \quad r = 1, \ldots, n,$$

and,

the likelihood L for the n independent experiments is

$$L = \prod_{r=1}^{n} L_r.$$

Let $I(\cdot|R)$ be the indicator function of R. Furthermore, define the function G_q by

$$G_q(z_1,\ldots,z_p) = \sum_{i=1}^{p} I(z_i|[S_q,S_{q+1}))$$

and 0 otherwise, and define the function $A_q(z_1,\ldots,z_p)$ by

$$A_q(z_1,\ldots,z_p) = \sum_{i=1}^{p} \{(S_{q+1}-S_q)\, I(z_i|[S_{q+1},T])$$

$$+ (z_i-S_q)\, I(z_i|[S_q,S_{q+1}))\}.$$

Let $Y_{ir} = X_{ir}$, $r = 1,\ldots,n$, $i = 1,\ldots,K_r-1$, and $Y_{K_r,r} = T - \sum_{i=1}^{K_r-1} X_{ir}$, $r = 1,\ldots,n$. Denote by G_{rq}, $q=0,\ldots,t-1$, the function $G_q(Y_{1r},\ldots,Y_{K_r-1,r})$, which is the number of failures in the r-th experiment which lie in $[S_q, S_{q+1})$. Also denote by A_{rq}, $q = 0,\ldots, t-1$, the function $A_q(Y_{ir},\ldots,Y_{K_r,r})$, which is the total time on test for the r-th experiment over $[S_q, S_{q+1})$.

Observe, too, that if $r(x) = f(x)/(1-F(x)]$, $x\varepsilon[0,T)$, then $F(x) = 1-\exp\{-\int_0^x r(y)dy\}$ and $f(x) = r(x)\exp\{-\int_0^x r(y)dy\}$ for $x\varepsilon[0,T)$.

The next lemma will allow us to easily find the values of the λ_q, $q = 0,\ldots, t-1$, say $\hat{\lambda}_{nq}$, $q = 0,\ldots, t-1$, which maximize $L = \prod_{r=1}^{n} L_r$, without the restriction that $\hat{\lambda}_{n0} \leq \hat{\lambda}_{n1} \leq \cdots \leq \hat{\lambda}_{n(t-1)}$. We call $\hat{\lambda}_{nq}$ the "naive" MLE of λ_q. (The term "unrestricted" MLE is also used in the literature.)

Lemma 4.1

The likelihood L_r for the r-th experiment may be written as

$$L_r = \prod_{j=0}^{t-1} \lambda_j^{G_{rj}} \exp(-\lambda_j A_{rj}), \quad r = 1,\ldots,n. \tag{4.1}$$

Proof:

First note that

$$r(x) = \sum_{j=0}^{t-1} \lambda_j \, I(x|[S_j,S_{j+1})),$$

for $x\varepsilon(0,T)$. Thus, L_r may be written as

$$L_r = \prod_{i=1}^{K_r-1} \left(\sum_{j=0}^{t-1} \lambda_j \, I(Y_{ir}|[S_j,S_{j+1})) \right)$$

$$\prod_{i=1}^{K_r} \exp\Big(-\sum_{j=0}^{t-1} \lambda_j \{(S_{j+1}-S_j) \, I(Y_{ir}|[S_{j+1},T])$$

$$+ (Y_{ir}-S_j) \, I(Y_{ir}|[S_j,S_{j+1}))\}\Big).$$

Now observe that

$$\prod_{j=0}^{t-1} \lambda_j^{G_{rj}} = \prod_{i=1}^{K_r-1} \left(\sum_{j=0}^{t-1} \lambda_j I(Y_{ir}[S_j,S_{j+1})) \right).$$

Equation (4.1) follows.

The following corollary gives the naive MLE $\hat{\lambda}_{nq}$ of λ_q, $q = 0,\ldots, t-1$.

Corollary 4.2

The maximum of $L = \prod_{r=1}^{n} L_r$ is obtained if $\lambda_q = \hat{\lambda}_{nq}$, $q = 0, \ldots, t-1$, where

$$\hat{\lambda}_{nq} = \frac{\sum_{r=1}^{n} G_{rq}}{\sum_{r=1}^{n} A_{rq}} \qquad \text{if } \sum_{r=1}^{n} A_{rq} \neq 0,$$

and

$$\lambda_{nq} = 0 \qquad \text{if } \sum_{r=1}^{n} A_{rq} = 0.$$

Proof:

If $\sum_{r=1}^{n} A_{rq} \neq 0$, the result follows directly from (4.1).

Also, $\sum_{r=1}^{n} A_{rq} = 0$ implies that $\sum_{r=1}^{n} G_{rq} = 0$. Since $\lambda_q \geq 0$ we define 0^0 to equal 1 and thus take $\hat{\lambda}_{nq} = 0$ if $\sum_{r=1}^{n} A_{rq} = 0$. This will maximize L.

Strong Consistency of $\hat{\lambda}_{nq}$

The next theorem will be used to show strong consistency of $\hat{\lambda}_{nq}$, $q = 0, \ldots, t-1$, given in Corollary 4.2.

Theorem 4.3

Let H be a c.d.f. such that $H(0) = 0$, and for u and v where $0 < u < v$, $H(v^-) - H(u^-) > 0$. Let $X_1, X_2, \ldots,$ be random variables with c.d.f. H, and let K be the stopping variable defined as the first integer such that

$$\sum_{i=1}^{K} X_i \geq T.$$

In addition suppose that H is absolutely continuous on [u, v) with p.d.f. h and

$$h(x)/[1-H(x)] = \lambda,$$

$x \in [u, v)$. Then

$$\frac{E\left(\sum_{i=1}^{K-1} I(X_i|[u, v))\right)}{E\sum_{i=1}^{K}\{(v-u)\, I(Y_i|[v, \infty)) + (Y_i-u)\, I(Y_i|[u, v))\}} = \lambda \qquad (4.2)$$

Proof

See Crow and Shimi (1971)

Theorem 4.4

The naive MLE $\hat{\lambda}_{nq}$ is a strongly consistent estimator of λ_q, $q = 0, \ldots, t-1$, as $n \to \infty$.

Proof:

If $\lambda_q = 0$, then $F(S_{q+1}) = 0$ and, hence, $\sum_{r=1}^{n} G_{rq} = 0$ a.s.

Thus $\hat{\lambda}_{nq} = 0$.

If $\lambda_q > 0$, then

$$\Big(\sum_{r=1}^{n} G_{rq}\Big)\Big/\Big(\sum_{r=1}^{n} A_{rq}\Big) \longrightarrow \frac{E(G_{1q})}{E(A_{1q})} = \lambda_q \text{ a.s. as } n \to \infty,$$

by the strong law of large numbers and (4.2).

The MLE of the λ_q

In the next theorem we will find the values of the λ_q, $q = 0,\ldots, t-1$, say $\tilde{\lambda}_{nq}$, which will maximize L under the restriction that $\tilde{\lambda}_{n0} \leq \tilde{\lambda}_{n1} \leq \ldots \leq \tilde{\lambda}_{n(t-1)}$. It will, also, be shown that $\tilde{\lambda}_{nq}$ is a strongly consistent estimator of λ_q.

Theorem 4.5

The MLE $\tilde{\lambda}_{nq}$ of λ_q, $q = 0,\ldots, t-1$, which maximizes $L = \prod_{r=1}^{n} L_r$ under the restriction that $\tilde{\lambda}_{n0} \leq \tilde{\lambda}_{n1} \leq \ldots \leq \tilde{\lambda}_{n(t-1)}$ is given by

$$\lambda_{nq} = \min_{v \geq q} \max_{u \leq q} \frac{\sum_{r=1}^{n} \sum_{d=u}^{v} G_{rd}}{\sum_{r=1}^{n} \sum_{d=u}^{v} A_{rd}} \quad . \tag{4.3}$$

Proof:

From Lemma (4.1)

$$L = \prod_{j=0}^{t-1} \lambda_j^{\sum_{r=1}^{n} G_{rj}} \exp\Big(-\lambda_j \sum_{r=1}^{n} A_{rj}\Big).$$

Applying the results of Brunk (1958) yields (4.3).

Remark 4.6

Brunk (1958), page 447, explains a method for determining $\tilde{\lambda}_{nq}$. Let $\hat{\lambda}_{nq}$ be the naive MLE of λ_q given in Corollary 4.2. If $\hat{\lambda}_{n0} \leq \hat{\lambda}_{n1} \leq \cdots \leq \hat{\lambda}_{n(t-1)}$, then $\tilde{\lambda}_{nq} = \hat{\lambda}_{nq}$. If for some i, $\hat{\lambda}_{ni} > \hat{\lambda}_{n(i+1)}$, then replace $\hat{\lambda}_{ni}$ and $\hat{\lambda}_{n(i+1)}$ by

$$[\sum_{r=1}^{n} (G_{ri} + G_{r(i+1)})]/[\sum_{r=1}^{n} (A_{ri} + A_{r(i+1)})].$$

If a reversal still exists, replace by appropriate averages. That is, if

$$[\sum_{r=1}^{n} (G_{ri} + G_{r(i+1)})]/[\sum_{r=1}^{n} (A_{ri} + A_{r(i+1)})] > \hat{\lambda}_{n(i+2)},$$

then replace $\hat{\lambda}_{ni}$, $\hat{\lambda}_{n(i+1)}$, and $\hat{\lambda}_{n(i+2)}$, by

$$[\sum_{r=1}^{n} (G_{ri} + G_{r(i+1)} + G_{r(i+1)})]/[\sum_{r=1}^{n} (A_{ri} + A_{r(i+1)} + A_{r(i+2)}].$$

Continue averaging whenever there is a reversal. This will yield a monotone increasing sequence, $\tilde{\lambda}_{n0} \leq \tilde{\lambda}_{n1} \leq \cdots \leq \tilde{\lambda}_{n(t-1)}$, which are the MLE's of the λ_q's subject to $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{t-1}$. We now have

Theorem 4.7

The estimates $\tilde{\lambda}_{nq}$ of λ_q, $q = 0,\ldots, t-1$, given in Theorem 3.5 are strongly consistent as $n \to \infty$.

Proof:

This is immediate from Remark 4.6 and the fact that the $\hat{\lambda}_{nq}$'s are strongly consistent estimators.

Asymptotic Normality of the Estimators

Without any loss of generality denote by λ_i, $i = 1,2,\ldots,m$, the set of all λ_i, $i = 0,1,\ldots, t-1$ such that $\lambda_i \neq 0$. In the following the asymptotic normality of the vector $(\hat{\lambda}_{n1},\ldots,\hat{\lambda}_{nm})$, suitably normalized, is established. The normalizing sequence is determined from the experimental outcomes and λ_i, $i = 1,\ldots,m$. Furthermore, it is shown that the dispersion matrix of the limiting distribution is the $m \times m$ identity matrix I.

Let

$$T_{nj} = \sum_{i=1}^{n} (G_{ij} - E(G_{ij}))/n, \quad j = 1,2,\ldots,m, \tag{4.4}$$

and

$$U_{nj} = \sum_{i=1}^{n} (A_{ij} - E(A_{ij}))/n, \quad j = 1,2,\ldots,m. \tag{4.5}$$

Define

$$Y_{nj} = \begin{cases} T_{nj} & j = 1,2,\ldots,m \\ U_{n(j-m)} & j = m+1,\ldots,2m \end{cases} \tag{4.6}$$

and $\Sigma = (\sigma_{ij})$ is a $2m \times 2m$ matrix, where

$$\sigma_{ij} = \mathrm{Cov}(Y_{1i}, Y_{1j}),\ i, j = 1,2,\ldots,2m.$$

By the multivariate central limit theorem, it follows that

$$\sqrt{n}\,(Y_{n1}, Y_{n2},\ldots,Y_{n,2m}) \text{ is AN } (\underline{0},\Sigma).$$

We shall use the following theorem given in Rao (1968, page 322), and we state it here for easy reference. "Let $\underline{T}_n$ be a k-dimensional statistic $(T_{1n},\ldots,T_{kn})$ with the asymptotic distribution of $\sqrt{n}\,((T_{1n}-\theta_1),\ldots,(T_{kn}-\theta_k))$ being k-variate normal with mean zero and dispersion matrix $\Sigma = (\sigma_{ij})$. Let $f_1,\ldots,f_q$ be q functions of k variables and each f_i be totally differentiable. Then the asymptotic distribution of $\sqrt{n}\left(f_i(t_{1n},\ldots,T_{kn}) - f_i(\theta_1,\ldots,\theta_k)\right)$, $i = 1,2,\ldots,q$, is q-variate normal with mean zero and dispersion matrix $\Gamma\,\Sigma\,\Gamma'$, where $\Gamma = (\partial f_i/\partial\theta_j)$."

Let $f_i(y_1,y_2,\ldots,y_{2m})$ be the real-valued function of 2m variables defined by

$$f_i(y_1,\ldots,y_{2m}) = y_i - \lambda_i y_{i+m},\ i = 1,2,\ldots,m. \tag{4.7}$$

Then

$$\partial f_i/\partial y_j = \begin{cases} 1 & \text{if } j = i \\ -\lambda_i & \text{if } j = i + m \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Therefore, one can show that $\Gamma \Sigma \Gamma' = (\tau_{ij})$, where

$$\tau_{ij} = \mathrm{Var}(G_{1i} - \lambda_i A_{1i}), \quad \text{if } i = j,$$

and

$$\tau_{ij} = \mathrm{Cov}(G_{1i}, G_{1j}) - \lambda_i \mathrm{Cov}(A_{1i}, G_{1j}) - \lambda_j \mathrm{Cov}(G_{1i}, A_{1j})$$

$$+ \lambda_i \lambda_j \mathrm{Cov}(A_{1i}, A_{1j}), \quad \text{if } i \neq j.$$

Hence, by the theorem mentioned above,

$$\sqrt{n} \left(\sum_{r=1}^{n} \frac{G_{r1} - \lambda_1 A_{r1}}{n}, \ldots, \sum_{r=1}^{n} \frac{G_{rm} - \lambda_m A_{rm}}{n} \right)$$

is AN $(\underline{0}, \Gamma \Sigma \Gamma')$.

Let $A_i = E(A_{1i})$, $i = 1,2,\ldots,m$. Note that $\lambda_i \neq 0$ implies $A_i \neq 0$. One can see then that the asymptotic distribution of

$$\underline{X}_n = \sqrt{n} \left(\sum_{r=1}^{n} \frac{G_{r1} - \lambda_1 A_{r1}}{nA_1}, \ldots, \sum_{r=1}^{n} \frac{G_{rm} - \lambda_m A_{rm}}{nA_n} \right)$$

is $N(\underline{0}, \Sigma_1)$, where $\Sigma_1 = (\tau_{ij}/A_i A_j)$.

Since $\frac{1}{n} \sum_{r=1}^{n} A_{ri} \to A_i$ a.s., one can, also, show that

$$\underline{Y}_n = \sqrt{n} \left(\sum_{r=1}^{n} \frac{G_{r1} - \lambda_1 A_{r1}}{\sum_{r=1}^{n} A_{r1}}, \ldots, \sum_{r=1}^{n} \frac{G_{rm} - \lambda_m A_{rm}}{\sum_{r=1}^{n} A_{rm}} \right)$$

is AN $(\underline{0},\Sigma_1)$. But since

$$\hat{\lambda}_i = \sum_{r=1}^{n} G_{ri} / \sum_{r=1}^{n} A_{ri}, \quad i = 1,2,\ldots,m,$$

it follows that $\sqrt{n}\ (\hat{\lambda}_{n1}-\lambda_1,\ldots,\hat{\lambda}_{nm}-\lambda_m)$ is AN $(\underline{0},\Sigma_1)$.

Let $D_i = \tau_{ij}/A_i^2$, $i = 1,2,\ldots,m$. Then

$$\sqrt{n}\ \left(\frac{\hat{\lambda}_{n1}-\lambda_1}{\sqrt{D_1}},\ldots,\frac{\hat{\lambda}_{nm}-\lambda_m}{\sqrt{D_m}}\right)$$

is AN$(\underline{0},\Sigma_2)$, where

$$\Sigma_2 = (\delta_{ij}) = (\tau_{ij}/A_iA_j(D_iD_j)^{\frac{1}{2}}). \tag{4.9}$$

Note that $\delta_{ij} = 1$ if $i = j$, and

$$D_i = \frac{1}{A_i^2}\ \mathrm{Var}(G_{1i}-\lambda_iA_{1i}), \quad i = 1,2,\ldots,m. \tag{4.10}$$

Observe, also, that $E(G_{1i}) = \lambda_iE(A_{1i})$, $i=1,2,\ldots,m$, by equation (4.2), Now, assuming that $\lambda_i \neq 0$ and $\lambda_s \neq 0$, this implies

$$0 = \frac{\partial}{\partial\lambda_s} \sum_{k=1}^{\infty} \int\ldots\int \left\{\left[\frac{1}{\lambda_i} G_i(x_1,\ldots,x_{k-1}) - A_i(x_1,\ldots,x_{k-1}, T - \sum_{\ell=1}^{k-1} x_\ell)\right] \prod_{j=0}^{t-1} \lambda_j^{G_j} \exp(-\lambda_jA_j) \prod_{i=1}^{k-1} dx_i.\right.$$

Thus, if i = s,

$$0 = \frac{1}{\lambda_i^2} E(G_{1i})^2 - \frac{1}{\lambda_i^2} E(G_{1i}) - \frac{2}{\lambda_i} E(G_{1i}-A_{1i}) + E(A_{1i})^2.$$

Using (4.2) again gives

$$E(G_{1i}) = \mathrm{Var}(G_{1i}) + \lambda_i^2 \mathrm{Var}(A_{1i}) - 2\lambda_i \mathrm{Cov}(G_{1i},A_{1i})$$

$$= \mathrm{Var}(G_{1i}-\lambda_i A_{1i}). \tag{4.11}$$

If $i \neq s$, then

$$0 = \sum_{k=1}^{\infty} \int\cdots\int \{\frac{1}{\lambda_i} G_i - A_i\} \{\frac{1}{\lambda_s} G_s - A_s\}$$

$$\prod_{j=0}^{t-1} \frac{\lambda_j^{G_j}}{} \exp(-\lambda_j A_j) \prod_{i=1}^{k-1} dx_i,$$

and this gives

$$0 = E(G_{1i}G_{1s}) - \lambda_s E(G_{1i}A_{1s}) - \lambda_i E(A_{1i}G_{1s})$$

$$+ \lambda_i\lambda_s E(A_{1i}A_{1s}).$$

Hence,

$$\lambda_s E(G_{1i})E(A_{1s}) + \lambda_i E(A_{1i})E(G_{1s}) - E(G_{1i})E(G_{1s})$$

$$- \lambda_i\lambda_s E(A_{1i})E(A_{1s}) = \mathrm{Cov}(G_{1i},G_{1s}) - \lambda_s \mathrm{Cov}(G_{1i},A_{1s})$$

$$- \lambda_i \mathrm{Cov}(A_{1i},G_{1s}) + \lambda_i\lambda_s \mathrm{Cov}(A_{1i},A_{1s}). \tag{4.12}$$

Since $E(G_{l\ell}) = \lambda_\ell E(A_{l\ell})$, it follows that the left-hand side of (4.12) is equal to zero and, hence, $\Sigma_2 = I$. Thus,

$$\sqrt{n}\left(\frac{\hat{\lambda}_{n1}-\lambda_1}{\sqrt{D_1}},\ldots,\frac{\hat{\lambda}_{nm}-\lambda_m}{\sqrt{D_m}}\right)$$

is AN $(\underline{0},I)$. Note that if $\lambda_1 < \ldots < \lambda_m$ then $\hat{\lambda}_{ni} = \tilde{\lambda}_i$ a.s. for $n \geq n_0$, say. Thus,

$$\sqrt{n}\left(\frac{\tilde{\lambda}_{n1}-\lambda_1}{\sqrt{D_1}},\ldots,\frac{\tilde{\lambda}_{nm}-\lambda_m}{\sqrt{D_m}}\right)$$

is AN $(\underline{0},I)$. By (4.11)

$$D_i = \frac{1}{A_i^2}\operatorname{Var}(G_{1i}-\lambda_i A_{1i}) = \frac{E(G_{1i})}{A_i^2}\,.$$

Using (4.2), this gives

$$D_i = \frac{\lambda_i^2}{E(G_{1i})}\,.$$

Note that since $\frac{1}{n}\sum_{r=1}^{n} G_{ri} \to E(G_{1i})$ a.s., then $\dfrac{\lambda_i^2}{\frac{1}{n}\sum_{r=1}^{n} G_{ri}} \to D_i$ a.s.

The next result is immediate.

Theorem 4.8

The asymptotic distribution of

$$\left((\hat{\lambda}_{n1}-\lambda_1)\Big/\sqrt{\frac{\lambda_1^2}{\sum_{r=1}^{n} G_{r1}}},\ldots,(\hat{\lambda}_{nm}-\lambda_m)\Big/\sqrt{\frac{\lambda_m^2}{\sum_{r=1}^{n} G_{rm}}}\right) \text{ is } N(\underline{0},\, I) \text{ as } n \to \infty.$$

As before, if $\lambda_1 < \ldots < \lambda_m$, then the same result holds if the $\hat{\lambda}_{ni}$ are replaced by $\tilde{\lambda}_{ni}$.

Remark 4.9

If one assumes that $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{t-1}$ then the likelihood $L = \prod_{r=1}^{n} L_r$ is the same and hence the naive MLE $\hat{\lambda}_{nq}$ of λ_q is the same and is consistent. The MLE $\tilde{\lambda}_{nq}$ may be easily found by applying again the results of Brunk (1958). Of course $\tilde{\lambda}_{nq}$ is also consistent in this case.

The reader is probably aware, at this point, that if t=1 then this class of distribution reduces to the class considered in the last section when $g(x) = x$. The MLE is equivalent in both situations.

5. NONPARAMETRIC ESTIMATION

Introduction

The concept of "failure rate" is a very important practical concept in reliability and has motivated several very useful classes of distributions, e.g., increasing failure rate (IFR) class, decreasing failure rate (DFR) class, u-shaped failure rate class. The failure rate $r(\cdot)$ of a distribution function F having derivative f is defined by

$$r(x) = f(x)/[1 - F(x)] \qquad \text{for } F(x) < 1$$

and

$$r(x) = \infty \qquad \text{for } F(x) = 1.$$

The estimation problem that we shall be concerned with in this section can be summarized in the following way. The life-time distribution of the items to be tested is assumed to have an increasing failure rate over the interval $[0,T)$, i.e., IFR on $[0,T)$. No other assumptions about the distribution or its failure rate is given outside that interval. The assumption of increasing failure rate can be changed to decreasing failure rate and the same results will follow with the obvious modifications. Data, of course, arise from Testing Plan A.

Let $[\alpha_F,\beta_F]$ be the support of the c.d.f. F. The notion of IFR on $[0,T)$ is made more precise by the following definition.

Definition 5.1

Let T be a fixed positive real number. A c.d.f. F, $F(0) = 0$, is said to be IFR (Increasing Failure Rate) on $[0,T)$ iff it satisfies one of the following conditions: (i) $-\log[1-F(x)]$ is convex on $[\alpha_F, \beta_F]$, $0 \leq \alpha_F \leq \beta_F \leq T$ and $F(\beta_F) = 1$ if $\beta_F < T$; or (ii) the part of the support of F in $[0,T]$ is empty.

Let $\mathcal{F} = \{F: F \text{ is IFR on } [0,T)\}$.

The following theorem is similar to a theorem concerning IFR distribution given by Marshall and Proschan (1965), and we shall omit its proof because of this similarity.

Theorem 5.2

Suppose $F \in \mathcal{F}$, $0 < Z \leq \beta_F$. Then F is absolutely continuous on $[0,Z)$. Note that F may take a jump at β_F if $\beta_F < T$.

Using the definition of failure rate and the above theorem one can very easily prove the following.

Theorem 5.3

(i) $F \in \mathcal{F}$ iff $r(\cdot)$ is nondecreasing on $[0,\beta_F)$, $0 \leq \alpha_F \leq \beta_F \leq T$, and $F(\beta_F) = 1$ if $\beta_F < T$.

(ii) The part of the support of F in $[0,T]$ is empty iff $r(x) = 0$ on $[0,T]$.

(iii) If $F \in \mathcal{F}$, then for $x \in [0, \beta_F)$

$F(x) = 1 - \exp(-R(x))$, and

$f(x) = r(x) \exp(-R(x))$, where

$$R(x) = \int_0^x r(y)dy.$$

The class $\bar{F}$ includes the usual class of IFR distributions. It is easy to show that there exists no sigma-finite measure relative to which all the distributions in $\bar{F}$ are absolutely continuous.

Since we are dealing with a nonparametric family of distributions for which there exists no sigma-finite measure relative to which all the measures induced by $\bar{F}$ are absolutely continuous, the usual concept of maximum likelihood estimation cannot be applied. The general definition of MLE due to Kiefer and Wolfowitz (1956) is used in this section to determine the MLE of the life-time distribution F over $[0,T)$, where $F \in \bar{F}$ and data arise from Testing Plan A. It is also shown that this MLE is strongly consistent as n, the number of original items, tends to infinity.

In this section let d(n) denote the total number of distinct failures in $[0,T)$ in the combined n experiments. Recalling the notation given in Section 2, observe that

$$0 \leq d(n) \leq \sum_{r=1}^{n} (K_r - 1).$$

Also, let $0 = Z_0 < Z_1 < \ldots < Z_{d(n)}$ be the ordered, distinct, failure times X_{jr}, $j = 1, \ldots, K_r - 1$, $r = 1, \ldots, n$. Finally, let p(n) be the number of $Y_{K_r r}$'s, $r = 1, \ldots, n$, strictly greater than $Z_{d(n)}$.

Maximum Likelihood Estimate

In the following we shall find the MLE of that part of a life-time distribution $F \in \bar{F}$ over the interval $[0,T)$ when data arise from

Testing Plan A. The following general definition of a maximum likelihood estimate is due to Kiefer and Wolfowitz (1956) and is needed to determine the MLE of $F \in \mathcal{F}$ for the two reasons mentioned earlier.

Definition 5.4

Let Ω be a sample space, $\mathcal{B}$ a σ-field on Ω, $\mathcal{P}$ a family of probability measures on $\mathcal{B}$ and Θ a set indexing the elements of $\mathcal{P}$ by $P(\cdot|\theta)$, $\theta \in \Theta$. Let $\underset{\sim}{X}$ be a random vector defined on Ω with distribution function determined by $P(\cdot|\theta_0)$, $\theta_0 \in \Theta$. If $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ denotes a random sample from $P(\cdot|\theta_0)$ then the MLE of θ_0 is $\hat{\theta}$ if $\hat{\theta} \in \Theta$ and

$$\sup_{\theta \in \Theta} \frac{\prod_{r=1}^{n} \frac{dP(\underset{\sim}{X}_r|\theta)}{d(P(\underset{\sim}{X}_r|\theta) + P(\underset{\sim}{X}_r|\hat{\theta}))}}{\prod_{r=1}^{n} \left\{1 - \frac{dP(\underset{\sim}{X}_r|\theta)}{d(P(\underset{\sim}{X}_r|\theta) + P(\underset{\sim}{X}_r|\hat{\theta}))}\right\}} = 1$$

where

$$\frac{dP(\cdot|\theta_1)}{d(P(\cdot|\theta_1) + P(\cdot|\theta_2))}$$

denotes the Radon-Nikodym derivative of $P(\cdot|\theta_1)$ with respect to $P(\cdot|\theta_1) + P(\cdot|\theta_2)$.

The Kiefer and Wolfowitz concept of MLE will now be considered within the framework of Testing Plan A and for life-time distributions $F \in \mathcal{F}$. Let $\Omega = \{\phi$, and all finite sequences of non-negative numbers whose sum is less than T$\}$, where ϕ is the empty set. Also, let $X_i = \{\underset{\sim}{x} \in \Omega$ which have exactly i elements$\}$, $i = 0, 1, \ldots$. Then $\Omega = \bigcup_{i=0}^{\infty} X_i$. We

define a set A to be measurable in Ω if and only if $A = \sum_{i=0}^{\infty} A_i$ and A_i is Borel measurable in X_i. Let $\mathcal{B}$ the σ-field of measurable sets in Ω.

For each $F \in \mathcal{F}$ we will define a probability measure $P(\cdot|F)$ on $\mathcal{B}$ and will denote the collection of all such measures by $\mathcal{P}$. These probability measures will be defined first on the Borel measurable sets of each X_i. Some preliminary notation is needed. Denote by $\lambda(\cdot|i, F)$ the product measure on R^i (Eucidean i-th space) induced by F, where $\lambda(\cdot|0, F)$ is defined to be one. Also, recall that $F(x^-) = \lim_{\varepsilon \to 0} F(x-\varepsilon)$, $\varepsilon > 0$, and products of the form $\prod_{j=1}^{0}$ and sums of the form $\sum_{j=1}^{0}$ are 1 and 0, respectively. For each Borel measurable set $A_i \subseteq X_i$ and $F \in \mathcal{F}$ define the measure $P(\cdot|F)$ to be

$$P(A_i|F) = \int_{A_i} \{1 - F([T - \sum_{j=1}^{i} x_j]^-)\} \, d\lambda(\underset{\sim}{x}|i,F).$$

For any $A \in \mathcal{B}$ we define $P(A|F)$ to be

$$P(A|F) = \sum_{i=0}^{\infty} P(A_i|F)$$

where $A_i = A \cap X_i$.

This definition is motivated by the integrand of equation (2.7).

Note that for each $F \in \mathcal{F}$

$$P(\Omega|F) = P(\sum_{i=0}^{\infty} X_i|F) = \sum_{i=0}^{\infty} P(X_i|F)$$

$$= \sum_{i=0}^{\infty} \text{Prob}(K = i + 1|F) = 1.$$

Thus, for each $F \in \mathcal{F}$, $P(\cdot|F)$ is a probability measure on $\mathcal{B}$

The Kiefer and Wolfowitz concept of maximum likelihood estimate together with our definition of the measures $P(\cdot|F) \in \mathcal{P}$, $F \in \mathcal{F}$, yields the MLE $\tilde{F}_n$ of F on [0,T) described in the next theorem.

Let $I(\cdot|S)$ be the indicator function of S. Also, let $n_r(y)$ denote

$$\sum_{i=1}^{k_r} I(Y_{ir}|[y,\infty)).$$

Theorem 5.5

The MLE $\tilde{F}_n$ of F has failure rate $\tilde{r}_n$ where $\tilde{r}_n$ is constant over $[Z_q, Z_{q+1})$, $q = 0,\ldots,d(n)$, and

$$\tilde{r}_n(Z_q) = \min_{d(n)+1\geq v\geq q+1} \max_{0\leq u\leq q} \frac{\sum_{r=1}^{n} \sum_{j=1}^{K_r-1} I(X_{jr}|[Z_n, Z_v))}{\sum_{r=1}^{n} \int_{Z_u}^{Z_v} n_r(y)dy}. \qquad (5.3)$$

Proof:

The proof of this theorem follows in a straightforward manner from the Kiefer-Wolfowitz definition of MLE using the probability measures we introduced above and Brunk's (1958) results.

Remark 5.6

We will now give a useful method for determining $\tilde{r}_n$. Let T_{qn} be the time on test over $[Z_q, Z_{q+1})$ (i.e. $T_{qn} = \sum_{r=1}^{n} \int_{Z_q}^{Z_{q+1}} n_r(y)dy$), $q = 0,\ldots,$ $d(n)$. If $(T_{0n})^{-1} \leq (T_{1n})^{-1} \leq \ldots \leq (T_{d(n)n})^{-1}$ then $\tilde{r}_n(Z_q) = (T_{qn})^{-1}$, $q = 0,\ldots,d(n)$. If for some i, $(T_{in})^{-1} > (T_{(i+1)n})^{-1}$ then replace $(T_{in})^{-1}$ and $(T_{(i+1)n})^{-1}$ by $2(T_{in} + T_{(i+1)n})^{-1}$.

If a reversal still exists, replace by appropriate averages. That is, if $2(T_{in} + T_{(i+1)n})^{-1} > (T_{(i+2)n})$, then replace $(T_{in})^{-1}$, $(T_{(i+1)n})^{-1}$ and

$(T_{(i+2)n})^{-1}$ by $3(T_{in} + T_{(i+1)n} + T_{(i+2)n})^{-1}$.

Continue averaging whenever there is a reversal. This will yield the monotone increasing sequence $\tilde{r}(Z_0) \leq \tilde{r}_n(Z_1) \leq \ldots \leq \tilde{r}_n(Z_{d(n)})$ given by (5.3).

Strong Consistency of $\hat{F}_n$

The main result of this section is that the MLE of F on [0,T) converges uniformly a.s. to F as the number of items put on test at time zero increases. To accomplish this we will prove a convergence theorem for $\tilde{r}_n(x)$, $x\varepsilon[0,T)$, $\tilde{r}_n(x)$ defined in Theorem 5.5. This result will allow us to easily prove the main result plus several corollaries. Furthermore, since the failure rate of a life-time distribution is an important practical concept, the convergence theorem for $\tilde{r}_n(x)$ is also a significant practical result.

We will need several theorems before we can prove the convergence theorem for $\tilde{r}_n(x)$. The next theorem involves rewriting $\tilde{r}_n(x)$, given Theorem 5.5, in a form we need to show consistency.

Let $R(u,v)$ denote $\sum_{r=1}^{n} \sum_{j=1}^{K_r-1} I(X_{jr} | [u,v))$ and $S(u,v)$ denote

$$\sum_{r=1}^{n} \int_{n}^{v} n_r(y)dy.$$

Theorem 5.7

Let $x\varepsilon[0,T)$ and

$$Z_n(x) = \max_{0 \leq i \leq d(n)} \{Z_i | Z_i \leq x\}.$$

Then

$$\tilde{r}_n(x) = \inf_{x \le v < T} \sup_{u < Z_n(x)} \frac{R(u,v)}{S(u,v)} . \qquad (5.4)$$

Proof:

Follows directly from Theorem 5.5.

To show consistency of $\tilde{r}_n$ we need the next two theorems. Let

$$M_n(u,v) = \frac{\sum_{r=1}^{n} \sum_{j=1}^{K_r - 1} I(X_{jr} | [u,v))}{\sum_{r=1}^{n} \int_u^v n_r(y)dy}, \quad 0 < u < v < T,$$

and let I_F be the intersection of the support of F with $[0,T]$.

Theorem 5.8

Let $0 \le u_0 \le v_0 \le T$ be fixed where, $0 \le u_0 < T$ if $I_F = \phi$, $0 \le u_0 < v_0 < \beta_F$ if $I_F = [\alpha_F, \beta_F]$. Then, as $n \to \infty$

i) $M_n(u_0,v)$ converges uniformly, a.s., in $v_0 \le v \le T$;

ii) $M_n(u,v_0)$ converges uniformly, a.s. in $0 \le u \le u_0$.

Proof:

Let $X_1, X_2, \ldots,$ be a sequence of independent, identically distributed random variables with c.d.f. F, $F(0) = 0$. Let N_1 be the first integer such that $\sum_{i=1}^{N_1} X_i \ge T$, N_2 the first integer such that $\sum_{i=N_1+1}^{N_2} X_i \ge T$, N_3 the first integer such that $\sum_{i=N_1+N_2+1}^{N_3} X_i \ge T$, and so on. Then $N_1, N_2, \ldots,$ is a sequence of i.i.d. random variables. Using the Glivenko-Cantelli theorem one may show that as $n \to \infty$

$$B_n(u,v) \longrightarrow F(v^-) - F(u^-), \text{ a.s., } -\infty < u < v < \infty, \qquad (5.5)$$

where

$$B_n(u,v) = \frac{\sum_{i=1}^{N(n)} I(X_i|[u,v))}{N(n)}, \quad N(n) = \sum_{r=1}^{n} N_r.$$

Also, using the strong law of large numbers and the Glivenko-Cantelli theorem it is easy to show that as $n \to \infty$

$$C_n(u,v) \longrightarrow \frac{1}{E(K_1)} [F_1(v^-) - F_1(u^-)] \text{ uniformly a.s.} \tag{5.6}$$

for $-\infty < u < v < \infty$, where

$$C_n(u,v) = \frac{\sum_{r=1}^{n} I(X_{K_r,r}|[u,v))}{A(n)}, \quad A(n) = \sum_{r=1}^{n} K_r,$$

and F_1 is the c.d.f. of $X_{K_1,1}$. We may conclude from (5.5) and (5.6) that as $n \to \infty$

$$D_n(u,v) = B_n(u,v) - C_n(u,v) \tag{5.7}$$

converges uniformly, a.s. for $-\infty < u < v < \infty$.

Similarly one may show that as $n \to \infty$

$$S_n(u,v) \text{ converges uniformly a.s. on } 0 \leq u < v \leq T \tag{5.8}$$

where

$$S_n(u,v) = \frac{\sum_{r=1}^{n} \int_u^v n_r(y)dy}{A(n)}.$$

Also, observe that for n_0 sufficiently large,

$$(S_n(u_0,v))^{-1} \text{ is uniformly bounded, a.s.,} \tag{5.9}$$

on $v_0 \leq v < T$, $n \geq n_0$,

$$D_n(u_0,v)) \text{ is uniformly bounded, a.s.} \tag{5.10}$$

on $v_0 \leq v < T$, $n \geq n_0$

$$(S_n(u,v_0))^{-1} \text{ is uniformly bounded, a.s.} \tag{5.11}$$

on $0 \leq u \leq u_0$, $n \geq n_0$, and

$$D_n(u,v_0) \text{ is uniformly bounded, a.s.} \tag{5.12}$$

on $0 \leq u \leq u_0$, $n \geq n_0$.

The proof is completed since (5.7) - (5.10) imply (i) and (5.7), (5.8), (5.11) and (5.12) imply (ii).

Theorem 5.9

Let F be IFR on $[0,T)$ with failure rate r on $[0,T)$. Then, for $0 \leq u < v < T$ fixed, where $0 \leq u < \beta$ if $I_F = [\alpha,\beta]$,

$$r(u) \leq \frac{E\left(\sum_{i=1}^{K-1} I(X_i|[u,v))\right)}{E\left(\int_u^v n(y)dy\right)} \leq r(v) \tag{5.13}$$

where $K = K_1$, $n(\cdot) = n_1(\cdot)$ and $X_i = X_{i1}$, $i = 1,2,\ldots$.

Proof:

If $I_F = \phi$ then F has failure rate 0 on [0,T) and, thus, (5.13) follows. If $I_F = \{\beta\}$ then $F(\beta) = 1$. Consequently, $r(x) = \infty$, $x \geq \beta$, and $r(x) = 0$, $x < \beta$. Also,

$$E\left(\sum_{i=1}^{K-1} I(X_i | [u,v))\right) = 0 \text{ for } u < v \leq \beta$$

and

$$E\left(\sum_{i=1}^{K-1} I(X_i | [u,v))\right) \geq 1 \text{ for } u < \beta < v.$$

Further,

$$E\left(\int_u^v n(y)dy\right) > 0 \text{ for } u < \beta.$$

Thus, (5.13) easily follows when $I_F = \{\beta\}$.

Now, assume $I_F = [\alpha,\beta]$, $0 \leq \alpha < \beta \leq T$. Also, recall that by Theorem 4.3, we know that if H is a c.d.f. with failure rate constant, say, λ, on [a,b), then

$$\frac{E_H\left(\sum_{i=1}^{K-1} I(X_i | [a,b))\right)}{E_H\left(\int_a^b n(y)dy\right)} = \lambda. \tag{5.14}$$

Case 1.

$$0 < u < v < \beta.$$

If F has a nondecreasing step-function failure rate on [u,v) then (5.13) holds by a simple application of (5.14). To prove that

(5.13) holds in general for this case, let $r_n(x)$, $n = 1,2,\ldots$, $x\epsilon[0,\beta)$ be a sequence of real valued functions such that $r_n(x) = r(x)$, $x\epsilon[0,u)$, $r_n(x)$ is a nondecreasing step-function on $[u,v)$ and $r_n(x) \uparrow r(x)$ on $[u,v)$. Note that $r_n(x) \leq r(x) \leq r(v) < \infty$. Thus, by the Lebesgue Dominated Convergence theorem, as $n \to \infty$

$$\int_0^y r_n(x)dx \to \int_0^y r(x)dx, \quad y \in [0,v).$$

Therefore,

$$F_n(y) \equiv 1 - \exp\{-\int_0^y r_n(x)dx\} \to 1 - \exp\{-\int_0^y r(x)dx\}$$

$$= F(y), \quad y \in [0,v), \text{ as } n \to \infty.$$

Let $F_n(y) = F(y)$, $y \geq v$. Then F_n, $n = 1,2,\ldots$, is absolutely continuous on $[0,v)$, continuous from the right on $[v,\infty)$, since F is, and $F_n(0) = 0$, $F_n(\infty) = 1$. Thus, F_n is a sequence of distribution functions, $F_n(y) \to F(y)$, $y \in (-\infty,\infty)$, as $n \to \infty$. Let

$$S_k = \{\sum_{i=1}^{k-1} x_i < T, \sum_{i=1}^{k} x_i \geq T\}, \quad k = 1,2,\ldots .$$

By the Helly-Bray theorem (Loeve (1963))

$$P[K=k|F_n] = \int_{S_k} \prod_{i=1}^{k} dF_n(x_i) \to \int_{S_k} \prod_{i=1}^{k} dF(x_i)$$

$$= P[K=k|F], \quad n \to \infty, \quad k = 1,2,\ldots . \tag{5.15}$$

Let $p_n(k) = P[K=k|F_n]$, $n = 1,2,\ldots$, $k = 1,2,\ldots$, and

$p(k) = P[K=k|F]$, $k = 1,2,\ldots$. By Rao ((1968), page 106) and (5.15)

$$\sum_{k=0}^{\infty} |p_n(k) - p(k)| \to 0, \; n \to \infty. \tag{5.16}$$

Now, note that

$$\sum_{i=1}^{K-1} I(X_i|[u,v)) \leq [\frac{T}{u}], \text{ a.s., } u > 0 \tag{5.17}$$

and

$$n(y) \leq [\frac{T}{y}], \text{ a.s., } y > 0 \tag{5.18}$$

where $[x]$ denotes the largest integer less than or equal to x.

Thus, since $u > 0$ and (5.16) holds

$$|E_{F_n}(\sum_{i=1}^{K-1} I(X_i|[u,v))) - E_F(\sum_{i=1}^{K-1} I(X_i|[u,v)))|$$

$$= |\sum_{k=1}^{\infty} \int_{A_k} \sum_{i=1}^{k-1} I(x_i|[u,v)) \prod_{j=1}^{k} dF_n(x_i)$$

$$- \sum_{k=1}^{\infty} \int_{A_k} \sum_{i=1}^{k-1} I(x_i|[u,v)) \prod_{j=1}^{k} dF(x_i)|$$

$$\leq [\frac{T}{u}] \sum_{k=1}^{\infty} |p_n(k) - p(k)| \to 0, \text{ as } n \to \infty.$$

Also,

$$\int_u^v n(y)dy \leq (v-u)\, n(u) \leq (v-u)[\frac{T}{u}]$$

by (5.16). Hence,

$$E_{F_n}\left(\sum_{i=1}^{K-1} I(X_i|[u,v)\right) \to E_F\left(\sum_{i=1}^{K-1} I(X_i|[u,v)\right) \tag{5.19}$$

and

$$E_{F_n}\left(\int_u^v n(y)dy\right) \to E_F\left(\int_u^v n(y)dy\right). \tag{5.20}$$

However, from (5.14) it follows that

$$r_n(y) \le \frac{E_{F_n}\left(\sum_{i=1}^{K-1} I(X_i|[u,v)\right)}{E_{F_n}\left(\int_u^v n(y)dy\right)} \le r_n(v).$$

Taking limits, and using (5.19) and (5.20) gives (5.13).

Case 2.

$$0 = u < v < \beta.$$

Inequalities (5.13) follows easily using the results of Case 1.

Also it is straightforward to use the results of Case 1 to prove (5.13) for

Case 3.

$$\beta < T, \ \beta \le v < T.$$

We now give the convergence theorem for the estimate $\tilde{r}_n(x)$ of $r(x)$, $0 \le x < T$.

Theorem 5.10

Let F be IFR on [0,T) with failure rate r on [0,T).

Then,

$$r(x_0^-) \leq \liminf \tilde{r}_n(x_0) \leq \limsup \tilde{r}_n(x_0) \leq r(x_0^+) \text{ a.s.}$$

for each $x_0 \in (0,T)$.

Proof:

Case 1.

$$I_F = \phi.$$

In this case $\tilde{r}_n(x) = 0$ a.s. for $0 \leq x < T$. Since $r(x) = 0$, $0 \leq x < T$, the result follows.

Case 2.

$$I_F = [\alpha,\beta].$$

Let $Z_n(x_0) = \max_{0\leq i\leq d(n)} \{Z_i | Z_i \leq x_0\}$. We will show the right-hand inequality first.

If $\beta < T$ and $\beta \leq x_0 < T$, then $r(x_0^+) = \infty$, since $F(\beta) = 1$.

Hence, assume $0 < x_0 < \beta \leq T$. Choose v_0, $x_0 < v_0 < \beta$. Then

$$\tilde{r}_n(x_0) = \inf_{x_0\leq v} \sup_{u<Z_n(x_0)} M_n(u,v)$$

$$\leq \sup_{u<Z_n(x_0)} M_n(u,v_0). \tag{5.21}$$

Let

$$M(a,b) = \frac{E\left(\sum_{j=1}^{K_1-1} I(X_{j1}|[a,b))\right)}{E\left(\int_a^b n_1(y)dy\right)}$$

Since $0 < x_0 < v_0$, we may apply Theorem 5.8 (ii) and conclude that as $n \to \infty$ $M_n(u,v_0)$ converges uniformly a.s. for $0 \leq u \leq x_0$. Thus, for arbitrary $\varepsilon > 0$ and $n \geq N(\varepsilon)$, say,

$$\tilde{r}_n(x_0) \leq \sup_{u<Z_n(x_0)} (M(u,v_0) + \varepsilon).$$

Since $u < \beta$ we may apply Theorem 5.9 and conclude that $\limsup \tilde{r}_n(x_0) \leq r(v_0) + \varepsilon$. This gives $\limsup \tilde{r}_n(x_0) \leq r(x_0^+)$ a.s. since $x_0 < v_0$ and the right-hand limits exist.

We will now show the left-hand inequality.

Case 2(a).

$$0 < \alpha \text{ and } x_0 \in (0,\alpha].$$

Since $r(x_0^-) = 0$ the left-hand inequality holds.

Case 2(b).

$$\beta < T \text{ and } \beta \leq x_0 < T.$$

If F takes a jump at β then with probability one $Z_n(x_0) = \beta$ for $n \geq N$, N sufficiently large. But this implies that $\tilde{r}_n(x) = \tilde{r}_n(\beta) = \infty$ for $n \geq N$, $\beta \leq x < T$. Thus, $\liminf \tilde{r}_n(x_0) = \infty$ and, hence, $\liminf \tilde{r}_n(x_0) \geq r(x_0^-)$.

If F does not take a jump at β then $r(\beta^-) = \infty$ and therefore as $n \to \infty$ $Z_n(x_0) \to \beta$ a.s.. Choose u_0, $0 < u_0 < \infty$. Then for N sufficiently large, $u_0 < Z_n(x_0) < \beta$, for $n \geq N$, and, thus,

$$\tilde{r}_n(x_0) = \inf_{x_0 \leq v} \sup_{u < Z_n(x_0)} M_n(u,v)$$

$$\geq \inf_{x_0 \leq v} M_n(u_0,v)$$

$$\geq \inf_{v_0 \leq v} M_n(u_0,v) \quad \text{for } u_0 < v_0 < \beta.$$

Apply Theorem 5.8 (i) and conclude that for arbitrary $\varepsilon > 0$, $N(\varepsilon)$ sufficiently large,

$$\tilde{r}_n(x_0) \geq \inf_{v_0 \leq v} (M(u_0,v)-\varepsilon) \text{ a.s., } n \geq N(\varepsilon).$$

By Theorem 5.9, $\tilde{r}_n(x_0) \geq r(u_0) - \varepsilon$, $n \geq N(\varepsilon)$. This gives $\liminf \tilde{r}_n(x_0) \geq r(u_0)$ a.s. for all $u_0 < \beta$. Letting $u_0 \to \beta^-$ gives $\liminf \tilde{r}_n(x_0) = \infty$ a.s.. Since $\tilde{r}_n(x_0^-) = \infty$ for $x_0 \geq \beta$, we have the desired result for Case 2(b).

Case 2(c).

$$\alpha < x_0 < \beta \leq T.$$

Choose u_0, $\alpha < u_0 < x_0$. Then for N large enough so that $u_0 < Z_n(x_0) < x_0$,

$$\tilde{r}_n(x_0) = \inf_{x_0 \le v} \sup_{u < Z_n(x_0)} M_n(u,v)$$

$$\geq \inf_{x_0 \le v} M_n(x_0,v).$$

Applying Theorems 5.8 and 5.9 in the usual manner gives $\liminf \tilde{r}_n(x_0) \geq r(u_0)$ a.s. for all $\alpha < u_0 < x_0$. The result follows.

This completes the proof.

The main result of this section is

Theorem 5.11

Let F be IFR on [0,T) with failure rate r on [0,T). Then $\tilde{F}_n(t) \to F(t)$ uniformly a.s. in $t \in [0,T)$, where

$$\tilde{F}_n(t) = 1 - \exp(-\int_0^t \tilde{r}_n(y)dy).$$

Proof:

Let I_F be the support of F on [0,T]. If $I_F = \phi$ the conclusion is clearly true. Note, also, that $\tilde{F}_n(0) = 0$ a.s.. Suppose then that $I_F = [\alpha,\beta]$. By Theorem 5.10 $\tilde{r}_n(t) \to r(t)$, $t \in [0,\beta)$ except possibly on a set of Lebesque measure zero. Let $t \in [0,\beta)$ and let $t \leq t_0 < \beta$ be a continuity point of r. For arbitrary $\varepsilon > 0$ and $N = N(t_0,\varepsilon)$ sufficiently large, $\tilde{r}_n(x) \leq \tilde{r}_n(t_0) \leq r(t_0) + \varepsilon$ for $x \in [0,t]$, $n \geq N$. Thus, by the Lebesgue Dominated Convergence theorem

$$\int_0^t \tilde{r}_n(z)dz \to \int_0^t r(z)dz \text{ a.s.} \tag{5.22}$$

Since

$$F(t) = 1 - \exp(-\int_0^t r(z)dz),\ t \in [0,\beta),$$

(5.22) implies that

$$\tilde{F}_n(t) \to F(t) \text{ a.s. } t \in [0,\beta). \tag{5.23}$$

Case 1.

F is continuous on $[0,T)$.

Since Testing Plan A and all related random variables are uneffected by the behavior of F on $[T,\infty)$, we may assume without any loss of generality that F is continuous on $[0,T]$.

Case 1a.

$$F(T) = 1,\ \beta = T.$$

Extend $\tilde{F}_n$ to $(-\infty,\infty)$ by defining $\tilde{F}_n(x) = 0$, $x < 0$, $\tilde{F}_n(x) = 1$, $x \geq T$. Then, by (5.23) as $n \to \infty$

$$\tilde{F}_n(t) \to F(t) \text{ a.s. } t \in (-\infty,\infty). \tag{5.24}$$

Note that $\tilde{F}_n$ is a distribution function. Since F is continuous on $(-\infty,\infty)$, $\tilde{F}_n(t) \to F(t)$ uniformly a.s. $t \in (-\infty,\infty)$, as $n \to \infty$ by Polya's theorem (Eisen, 1969).

Case 1b.

$$F(T) < 1,\ \beta = T.$$

Since $1-F(T^-) > 0$, there exists a.s. some $m = 1,2,\ldots$, such that $K_m = 1$. This implies that either $d(n) = 0$, or $p(n) > 1$ and $d(n) \geq 1$,

for $n \geq m$. In any case, $\tilde{r}_n(x) < \infty$ a.s. $x \in [0,T)$, for $n \geq m$. Therefore, $\tilde{F}_n(x) < 1$, $x \in [0,T)$, and hence, $\tilde{F}_n$ may be extended to $[0,T]$ in a continuous manner, for $n \geq m$. Since F is continuous, $\tilde{F}_n(T) \to F(T)$. Let

$$G_n(x) = \begin{cases} \dfrac{\tilde{F}_n(x)}{\tilde{F}_n(T)} & x \in [0,T] \\ 1 & x \in (T,\infty) \\ 0 & x \in (-\infty,0) \end{cases} \tag{5.25}$$

for $n \geq m$, and let

$$G(x) = \begin{cases} \dfrac{F(x)}{F(T)} & x \in [0,T] \\ 1 & x \in (T,\infty) \\ 0 & x \in (-\infty,0) \, . \end{cases} \tag{5.26}$$

Then G_n, $n = m, m+1, \ldots,$ G_n are distribution functions, G is continuous and $G_n(x) \to G(x)$ a.s. $x \in (-\infty,\infty)$. By Polya's theorem the convergence is uniform. Since $\tilde{F}_n(T)$ is bounded for $n = m, m+1, \ldots,$ this implies that $\tilde{F}_n(t) \to F(t)$ uniformly a.s. $t \in [0,T)$, as $n \to \infty$.

Case 1c.

$$\beta < T.$$

Let $\beta \leq x < T$ and $\varepsilon > 0$ be given. By the continuity of F there exists a $0 < z < \beta$ such that $1-F(z) \leq \varepsilon$, and by (5.24) there exists a $N = N(z,\varepsilon)$ such that $F(z) - \varepsilon \leq \tilde{F}_n(z)$, $n \geq N$. Hence, for $n \geq N$, $1 - 2\varepsilon \leq F(z) - \varepsilon \leq \tilde{F}_n(z) \leq \tilde{F}_n(x) \leq 1$. Therefore,

$\lim_{n \to \infty} \tilde{F}_n(x) \to F(x) = 1$ a.s. for $x \geq \beta$. Using (5.24) we have $\tilde{F}_n(x) \to F(x)$ a.s. for $x \in [0,T)$. Using Polya's theorem again we may conclude that $\tilde{F}_n(x) \to F(x)$ uniformly a.s. for $x \in [0,T)$, as $n \to \infty$.

Case 2.

F takes a jump on $[0,T)$.

Since F takes a jump on $[0,T)$ at β, it follows that with probability one $K_m = 1$, for some $m = 1,2,\ldots$. Thus, $\tilde{r}_n(t) = \infty$, $\beta \leq t < T$, $n \geq m$, which imples that $\tilde{F}_n(t) = 1$, $\beta \leq t < T$, $n \geq m$. Since $F(t) = 1$, $t \geq \beta$, we have

$$\tilde{F}_n(t) \to F(t) \text{ uniformly a.s. } t \in [\beta,T), \text{ as } n \to \infty. \tag{5.27}$$

We will now show that the convergence is uniform on $[0,T)$. For $n \geq m$, $\tilde{F}_n$ is a sequence of nondecreasing, bounded, continuous functions. Hence, they may be extended to $[0,\beta]$ in a fashion which will preserve continuity. Similarly, we may extend F to $[0,\beta]$ in a continuous manner. By (5.23), $\tilde{F}_n(t) \to F(t)$, $t \in [0,\beta)$, as $n \to \infty$. Let $F_n^*(\beta)$, $F^*(\beta)$, be the extended values of F_n and F for $n \geq m$. It is straightforward to show that $F_n^*(\beta) \to F^*(\beta)$ a.s. as $n \to \infty$. Now, let

$$H_n(x) = \begin{cases} \dfrac{\tilde{F}_n(x)}{F_n^*(\beta)} & 0 \leq x < \beta, \\ 1 & x \geq \beta, \\ 0 & x < 0 \end{cases}$$

$$H(x) = \begin{cases} \dfrac{F(x)}{F^*(\beta)} & 0 \leq x < \beta. \\ 1 & x \geq \beta, \\ 0 & x < 0 \end{cases}$$

for $n \geq m$. Note, then, for $n \geq m$, H_n, H are continuous distribution functions, and $H_n(x) \to H(x)$ a.s. Applying Polya's theorem again we may conclude that $H_n(x) \to H(x)$ uniformly a.s., $x\varepsilon(-\infty,\infty)$, as $n \to \infty$. Then

$$\tilde{F}_n(x) \to F(x) \text{ uniformly a.s. } x\varepsilon[0,\beta), \text{ as } n \to \infty. \tag{5.28}$$

Thus, (5.27) and (5.28) give the desired result. This completes the proof.

We now give two useful corollaries of Theorems 5.10 and 5.11.

Corollary 5.12

Let $S = [u,v]$ be a closed interval of continuity of r, $0 \leq u < v < T$. Then, $\tilde{r}_n(x) \to r(x)$ uniformly a.s. on S as $n \to \infty$.

Proof.

By Theorem 5.10

$$\tilde{r}_n(x) \to r(x) \text{ a.s. on } S \text{ as } n \to \infty. \tag{5.29}$$

Case 1.

$$r(u) = r(v) \geq 0$$

For N sufficiently large

$$r(u) - \varepsilon \leq \tilde{r}_n(u) \leq r(u) + \varepsilon$$

and

$$r(v) - \varepsilon \leq \tilde{r}_n(v) \leq r(v) + \varepsilon.$$

But

$$\tilde{r}_n(u) \leq \tilde{r}_n(x) \leq \tilde{r}_n(v).$$

The result follows.

Case 2.

$$r(v) > r(u).$$

Let

$$D_n(x) = \begin{cases} \dfrac{\tilde{r}_n(x) - \tilde{r}_n(u)}{\tilde{r}_n(v) - \tilde{r}_n(u)} & u \leq x \leq v \\ 0 & x < u \\ 1 & x > v \end{cases}$$

$n = 1,2,\ldots,$ and let

$$D(x) = \begin{cases} \dfrac{r(x) - r(u)}{r(v) - r(u)} & u \leq x \leq v \\ 0 & x < u \\ 1 & x > v \end{cases}$$

Note that D_n, D are distribution functions and D is continuous.

Applying Polya's theorem (Eisen (1969)) and (5.29) gives

$D_n(x) \to D(x)$ uniformly a.s. on $(-\infty,\infty)$ as $n \to \infty$. For $x \in S$, the sequence $\{r_n(x)\}$, $n \geq 1$ is ultimately uniformly bounded. This implies that $\tilde{r}_n(x) \to r(x)$ uniformly a.s. on S as $n \to \infty$. The proof is completed.

Corollary 5.13

Let $S = [u,v]$ be a closed interval of continuity of r, $0 \leq u < v < T$. Then,

$$\tilde{f}_n(x) \to f(x) \text{ uniformly a.s. on S as } n \to \infty$$

where $\tilde{f}_n(x) = \tilde{r}_n(x) \exp(-\int_0^x \tilde{r}_n(y)dy)$.

Proof.

The proof follows directly from Theorem 5.11 and Corollary (5.12).

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AN AGE REPLACEMENT FORMULA

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The basic idea behind this paper is that it may be only academic (if not pointless) to ponder the probability of an event unless one can weigh the consequences of whether or not the event occurs.

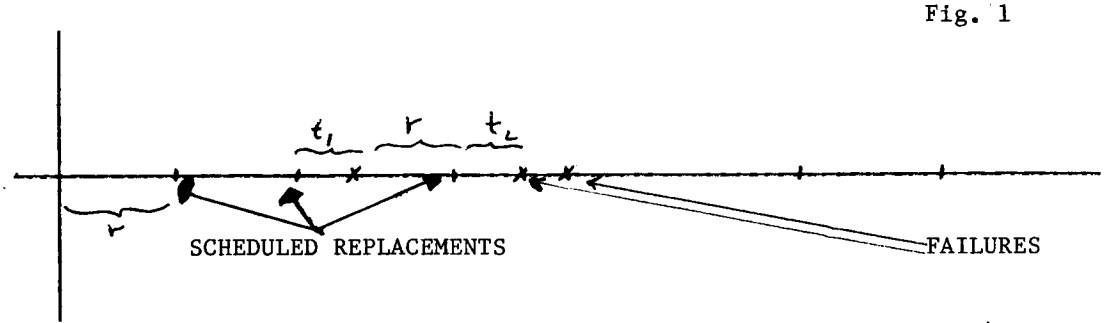
One might typically be asked, "What is the probability that a new part will survive its "assigned" mission life?" One might be given some data with which to estimate parameters and one might then obtain a lower confidence limit on this probability. How high should this probability be? If it is high, is that good? If it is low, is that bad? The answer is that regardless of its value, it is neither good nor bad - it is simply arbitrary.

The problem should be one of selecting a figure of merit for the part, or more precisely, a figure of merit for the replacement policy which might be followed with respect to the part.

Now, one could only want to replace a part before failure if extra costs[1] were incurred as a result of a failure of the part in service; i.e. if it were no more inconvenient to sustain a failure than to replace the part before failure, then the greatest use would be obtained from a series of parts by leaving each one in service until it failed.

Consider a part (of a system) having a failure density f(X); also consider the stochastic process in which n of these parts are used up one after the other such that a part is replaced either (1) when it fails (perhaps causing damage to associated parts) or (2) when it survives for a time r (the scheduled replacement time). A visualization of the process is given in figure (1).

Fig. 1



(the tacit assumption here is that the rest of the system is perfect)

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[1] extra damage, downtime, etc.

The density of the conditioned random variables t_i is

$$\frac{f(x)}{F(r)} \quad (t_i \leq r) \qquad \text{where } F(r) = \int_o^r f(x)dx$$

Let ℓ be the number of failures which take place during this binomial process. The expected value of ℓ is:

$$E(\ell) = np = nF(r)$$

Let the effectiveness of the series of parts be defined as the total time in service of the n parts:

$$T = \sum_{i=1}^{\ell} t_i + (n-\ell)r = S + (n-\ell)r$$

The expected value of the total service time is:

$$E(T) = E(\ell)E(t) + (n-E(\ell))r$$

$$= nF(r)E(t) + (n-nF(r))r$$

$$= n\{F(r)E(t) + (1-F(r))r\}$$

but

$$E(t) = \int_o^r u \frac{f(u)}{F(r)} du$$

$$= \frac{1}{F(r)} \int_o^r udF(u)$$

$$= \frac{1}{F(r)} (rF(r) - \int_o^r F(u)du)$$

$$= r - \frac{1}{F(r)} \int_o^r F(u)du.$$

Therefore:

$$E(T) = n\{F(r)(r - \frac{1}{F(r)} \int_o^r F(u)du) + r(1-F(r))\}$$

$$= n\{r - \int_o^r F(u)du\}$$

$$= n\{ \int_o^r du - \int_o^r F(u)du\}$$

$$= n \int_o^r [1-F(u)]du.$$

Let C_r and C_f be the costs of replacement prior to failure and replacement at failure respectively. The cost of failure C_f would, in general, be greater than C_r. The total cost and expected value thereof is:

$$C_T = \ell C_f + (n-\ell)C_r$$

$$= \ell(C_f - C_r) + nC_r$$

$$E(C_T) = nF(r)(C_f - C_r) + nC_r$$

$$= n\{F(r)(C_f - C_r) + C_r\}.$$

The figure of merit for the part and its associated replacement schedule may be defined as the expected ratio of total service time to total cost:

$$M_n(r) = E(\frac{T}{C_T})$$

for large n,

$$M_n(r) \to \frac{E(T)}{E(C_T)} = M_\infty(r)$$

$$M_n(r) = \frac{n \int_o^r [1-F(u)]du}{n\{F(r)(C_f - C_r) + C_r\}} .$$

Therefore:

$$M_\infty(r) = \frac{\int_o^r [1 - F(u)]du}{F(r)(C_f - C_r) + C_r} .$$

Since M_∞ is a function of r, the question naturally arises as to what value of r will maximize M_∞; this value of r would be called the optimal, long run replacement policy associated with the part.

An asymptotic formula for $M_n(r)$ will now be derived using the Taylor series method.

Since we are interested in the expected value of a quotient, we expand x/y in a Taylor series of ten terms and take the expected value thereof, obtaining:

$$E\left(\frac{x}{y}\right) = \frac{x_o}{y_o} - \frac{\mu_{11}(x,y)}{y_o^2} + \frac{x_o}{y_o^3}\,\mu_{02}(x,y) + \frac{\mu_{12}(x,y)}{y_o^3} - \frac{x_o}{y_o^4}\,\mu_{03}(x,y)$$

where

$x_o, y_o = E(x), E(y)$ respectively

and

$$\mu_{ij}(x,y) = E[(x-x_o)^i (y-y_o)^j].$$

Now in our context,

x = total service time

$$= T = \sum_{i=1}^{\ell} t_i + (n-\ell)r$$

$$= S + (n-\ell)r.$$

Therefore:

$$x_o = E(S) + r(n-E(\ell))$$

and $$x-x_o = S-E(S) - r(\ell-E(\ell))$$

and y = total cost

$$= C_T = \ell(C_f - C_r) + nC_r$$

$$= C\ell + nC_r.$$

Therefore:

$$y_o = CE(\ell) + nC_r$$

$$y-y_o = C(\ell-E(\ell)).$$

Now we obtain

$$\mu_{02}(x,y), \mu_{03}(x,y), \mu_{11}(x,y) \text{ and } \mu_{12}(x,y)$$

$\mu_{ok}(x,y)$:

$$(x-x_o)^o(y-y_o)^k = [C(\ell-E(\ell))]^k .$$

Therefore:

$$\mu_{ok}(x,y) = C^k E[(\ell-E(\ell))^k] = C^k \mu_{ko}(\ell,S)$$

$\mu_{1k}(x,y)$:

$$(x-x_o)(y-y_o)^k = [S-E(S) - r(\ell-E(\ell))]\,[C(\ell-E(\ell)]^k$$

$$= C^k(S-E(S))(\ell-E(\ell))^k - rC^k(\ell-E(\ell))^{k+1},$$

hence

$$\mu_{1k}(x,y) = C^k \mu_{k1}(\ell,S) - rC^k \mu_{k+1,0}(\ell,S)$$

therefore,

$$\mu_{02}(x,y) = C^2 \mu_{20}(\ell,S)$$

$$\mu_{03}(x,y) = C^3 \mu_{30}(\ell,S)$$

$$\mu_{11}(x,y) = C\mu_{11}(\ell,S) - rC\mu_{20}(\ell,S)$$

$$\mu_{12}(x,y) = C^2 \mu_{21}(\ell,S) - rC^2 \mu_{30}(\ell,S)$$

Now $\mu_{ko}(\ell,S)$ and $\mu_{k1}(\ell,S)$ must be obtained, but first, the joint density of ℓ and S is needed.

$$S = \sum_{i=1}^{\ell} t_i$$

$$P(S \in (s,s+ds),\ \ell=k) = h(s,k)ds$$

$$= P(S \in (s,s+ds) \mid \ell=k)P(\ell=k)$$

$$= P(S_k \in (s,s+ds))p_k$$

$$= g_k(s)ds \cdot p_k .$$

Therefore:

$$h(s,k) = p_k g_k(s)$$

where

$$S_k = \sum_{i=1}^{k} t_i$$

(k not a random variable) and g and g_k are the densities of t_i and S_k respectively.

Now, $E(S) = E(t)E(\ell)$

has been used earlier, but now it can be derived:

$$E(S) = \sum_{k} \int_{s} s p_k g_k(s) ds$$

$$= \sum_{k} p_k \int_{s} s g_k(s) ds$$

$$= \sum_{k} p_k E(S_k) = \sum_{k} p_k k E(t)$$

$$= E(t)E(\ell)$$

Now, $\mu_{20}(\ell,S)$ and $\mu_{30}(\ell,S)$ are just the second and third central moments of a binomially distributed random variable:

$$\mu_{20}(\ell,S) = np(1-p)$$

$$= nF(r)(1-F(r)) = na_1$$

and $$\mu_{30}(\ell,S) = np(1-p)(1-2p)$$

$$= nF(r)(1-F(r))(1-2F(r)) = na_2$$

Obtaining $\mu_{ml}(\ell,S)$:

$$\mu_{ml}(\ell,S) = E[(\ell-E(\ell))^m(S-E(S))]$$

$$= \sum_{ks}\int (k-E(\ell))^m(s-E(S))p_k g_k(s)ds$$

but

$$s-E(s) = s-E(S_k) + E(S_k) - E(S)$$

$$= s-E(S_k) + kE(t) - E(\ell)E(t)$$

$$= s-E(S_k)+E(t)(k-E(\ell)).$$

Therefore:

$$\mu_{ml} = \sum_{ks}\int (k-E(\ell))^m[s-E(S_k) + E(t)(k-E(\ell))]p_k g_k(s)ds$$

$$= \sum_{ks}\int \{(s-E(S_k))(k-E(\ell))^m + E(t)(k-E(\ell))^{m+1}\}p_k g_k(s)ds$$

$$= \sum_k p_k(k-E(\ell))^m \int_s (s-E(S_k))g_k(s)ds + E(t)\sum_k p_k(k-E(\ell))^{m+1}\int_s g_k(s)ds$$

$$= E(t)E[(\ell-E(\ell))^{m+1}] = E(t)\mu_{m+1,0}(\ell,S)$$

since $\int_s (s-E(s_k))g_k(s)ds = o$

and

$$\int_s g_k(s)ds = 1$$

therefore,

$$\mu_{11}(\ell,S) = E(t)\mu_{20}(\ell,S)$$

$$= na_1E(t)$$

$$\mu_{21}(\ell,S) = E(t)\mu_{30}(\ell,S)$$

$$= na_2E(t) \quad .$$

Therefore:

$$\mu_{02}(x,y) = C^2\mu_{20}(\ell,S) = na_1C^2 = nb_1$$

$$\mu_{03}(x,y) = C^3\mu_{30}(\ell,S) = na_2C^3 = nb_2$$

$$\mu_{11}(x,y) = C\mu_{11}(\ell,S) - rC\mu_{20}(\ell,S)$$

$$= Cna_1E(t) - rCna_1$$

$$= Cna_1(E(t) - r) = nb_3$$

$$\mu_{12}(x,y) = C^2\mu_{21}(\ell,S) - rC^2\mu_{30}(\ell,S)$$

$$= C^2na_2E(t) - rC^2na_2$$

$$= C^2na_2(E(t) - r) = nb_4 .$$

Recalling that

$$x_o = E(S) + r(n-E(\ell))$$

and

$$y_o = CE(\ell) + nC_r$$

one has:

$$x_o = E(\ell)E(t) + r(n-nF(r))$$

$$= nF(r)E(t) + nr(1-F(r))$$

$$= n[r+F(r)(E(t)-r)]$$

$$= nb_5$$

$$y_o = CnF(r) + nC_r = n[CF(r) + C_r]$$

$$= nb_6 .$$

Now,

E(x/y) may be evaluated:

$$E(\frac{x}{y}) \simeq \frac{x_o}{y_o} - \frac{\mu_{11}(x,y)}{y_o^2} + \frac{x_o}{y_o^3}\mu_{02}(x,y) + \frac{\mu_{12}(x,y)}{y_o^3} - \frac{x_o}{y_o^4}\mu_{03}(x,y)$$

$$E(\frac{x}{y}) \simeq \frac{nb_5}{nb_6} - \frac{nb_3}{n^2b_6^2} + \frac{nb_5}{n^3b_6^3} + \frac{nb_4}{n^3b_6^3} - \frac{nb_5 \cdot nb_2}{n^4b_6^4}$$

$$= \frac{b_5}{b_6} - \frac{b_3}{nb_6^2} + \frac{b_1b_5}{nb_6^3} + \frac{b_4}{n^2b_6^3} - \frac{b_2b_5}{n^2b_6^4} \quad .$$

Therefore:

$$M_n(r) \simeq \frac{b_5}{b^6} - \frac{1}{nb_6^2} \cdot (b_3 - \frac{b_1b_5}{b_6}) + \frac{1}{n^2b_6^3}(b_4 - \frac{b_2b_5}{b_6})$$

It is instructive to calculate $M_n(r)$ when the failure density is $\delta(x-\mu)$ where δ is the Dirac delta function. In this instance, the variance σ^2 is zero.

Doing this, one has:

$$M_n(r) = \begin{cases} \frac{r}{C_r} & , \quad r < \mu, \\ \frac{M}{C_f} & , \quad r > \mu. \end{cases}$$

Figure 2

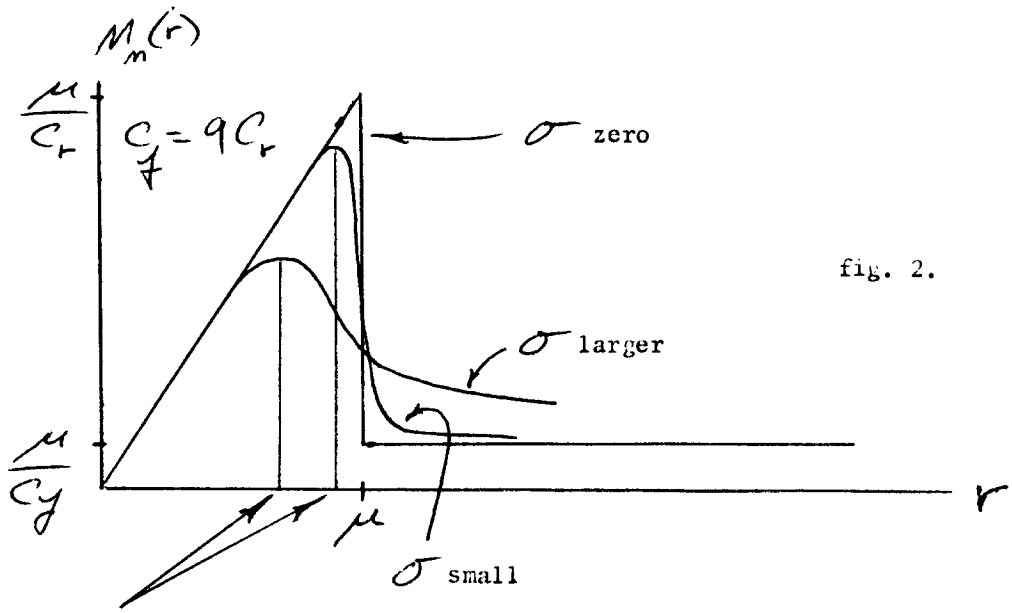

fig. 2.

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   Watervliet Arsenal Technical Report
   WVT-7040

A TECHNIQUE FOR OBTAINING A MEASURE OF INDUSTRIAL LEARNING AND LEVEL-OFF USING ECONOMIC PARAMETERS

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For years the aerospace industry has made use of what have been called "learning", "progress", "improvement", or "experience" curves to predict reductions in cost as the number of items produced increases. The learning process is a phenomenon that prevails in many industries and its existence has been verified by empirical data and controlled tests. Although there are several hypotheses on the exact manner in which the learning or cost reduction can occur, the basis of learning-curve theory is that each time the total quantity of items produced doubles, the cost per item is reduced to a constant percentage of its previous cost. The factors that account for the decline in unit cost as cumulative output increases are numerous and not completely understood. Those most commonly referenced are:

1. Job familiarization by workmen, which results from the repetition of manufacturing operations.

2. General improvement in tool coordination, shop organization, and engineering liaison.

3. Development of more efficiently produced subassemblies.

4. Development of more efficient parts-supply systems.

5. Development of more efficient tools.

6. Substitution of cast or forged components for machined components.

7. Improvement in overall management.

Psychological and Historical Perspective. The basis of the "learning or improvement" curve can be found in modern experimental psychology. The term used by the learning psycologists is "the continuity hypothesis". According to this hypothesis, learning is continuous and cumulative. Every reinforcement adds strength to the learning. One implication of the continuity hypothesis is that there is a regular progressive increase in the learning process. It would be logical to expect some decline in the rate of increase even though the learning continues. This hypothesis can be graphically represented by the Osgood Transfer Surface which is, in part, represented on the following page:

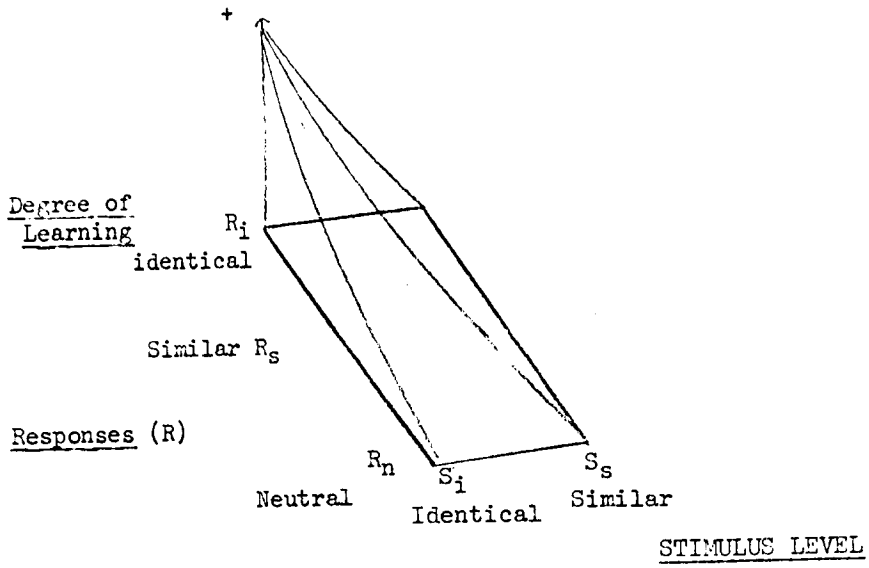
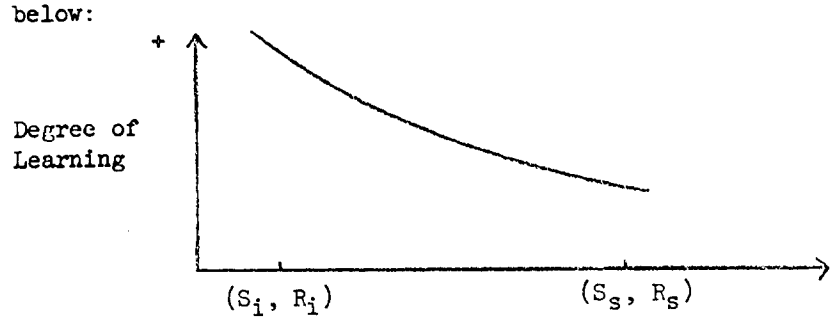


Figure (1)

Note that when the stimulus level is identical (the same manipulation needs to be performed: and the response is the same (the same manipulation is performed), then the "degree of learning" is a positive maximum. However, in the real world, the stimulus will vary (although be similar) and some appropriate and similar response will occur to meet these similar stimuli. What theoretically happens is that the "learner" (be it a person or a complex system) will move down the transfer surface given as Figure (1). The locus of the path is shown below:



Similar Stimuli (S) and Responses (R)

The curve represented above is commonly called "the learning curve". Bugelski (reference 1) points out that these curves drawn from individual learning data rarely look like these so-called textbook learning curves. Individual patterns tend to be erratic, with many ups-and-downs. It might be pointed out that in recent years psychologists have become

disenchanted with the notion of a learning curve. They find that there are too many variables influencing the process of learning. K. J. Hayes (in reference 1) says that if there is a typical curve, it is not like those published in earlier textbooks. The learning process depends upon what is being learned, the readiness of the learner, various individual difference factors, and such parameters as the drive state, incentive values, complexity of the stimulus situation and the features of the performance being measured.

In spite of these warnings of many psychologists, the learning curve has been used extensively by management as an emperical cost estimating relationship. The industrial application of the improvement curve can be traced to an article by Mr. T. P. Wright called "Factors Affecting the Cost of Airplanes." This article appeared in the February 1936 issue of The Journal of the Aeronautical Sciences.

General Equation. The emperical "improvement curve" equation is given by:

$$Y_c = ax^b \qquad (1)$$

where Y_c = cumulative average expenditure per unit (the average direct labor hours, or direct labor cost of all units including the given unit).

a = Initial expenditure. It is usually thought of as representing the first unit expenditure in production.

x = Cumulative number of units produced.

b = Characteristic exponent of the emperical curve.

Equation (1) represents the function as it would "usually" appear plotted on rectagular coordinates. Equation (1') is obtained by taking the logarithm of both sides of this equation:

$$\log Y_c = \log a + b \log x \qquad (1')$$

Where Y_c and "a" maintain their identity, but "b" is now called the characteristic slope of a linear equation. This equation can be conveniently fitted to data by the method of least squares.

Learning Plateau or Level-Off Considerations

The problem considered in this paper is to develop an objective means of determining a level-off quantity that depends on the characteristics of the improvement curve. That is, a procedure which would give the same value for level-off quantity regardless of who the analyst is. The development of the operational level-off is given below:

This concept of operational level-off depends on relative change. It is defined as follows:

Definition: x_1 is a level-off cumulative quantity if the next buy quantity Q results in a relative decrease of expenditure equal to some small fraction p.

The derivation of the level-off quantity x_1 follows from elementary calculus. The equation for the improvement curve is:

$$y = ax^b \tag{1}$$

then

$$dy = \left(\frac{dy}{dx}\right)dx \tag{2}$$

where

$$\frac{dy}{dx} = abx^{b-1}.$$

It can be seen that:

$\frac{dy}{y}$ = relative change in y. Rewriting equation (2) to account for relative change . . .

$$\frac{dy}{y} = \left(\frac{y'}{y}\right)dx \tag{3}$$

Substituting for y' and y in the right hand side of equation (3) . . .

$$\frac{dy}{y} = \left(\frac{ab\ x^{b-1}}{a\ x^b}\right)dx$$

and simplifying

$$\frac{dy}{y} = \left(\frac{b}{x}\right)dx \tag{4}$$

Let us now apply the results of equation (4) to an improvement curve situation.

Let 1. $dx \equiv \Delta x$. This is some average procurement buy quantity. For sake of notational brevity, call this Q.

2. $\frac{dy}{y} \equiv$ some appropriate relative change fraction representing a pre-defined "operational level-off". Again for sake of brevity, call this -p.

3. b ≡ characteristic slope of the improvement curve (i.e.; -b).

4. x_1 ≡ the cumulative quantity which satisfies the condition of (dy/y) under the constraints of b and dx. Since this is the operation level-off quantity, call this $LOQ_{(p, b)}$.

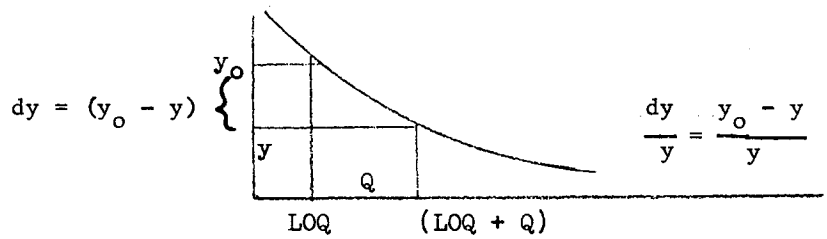
Equation (4) can now be rewritten as:

$$-p = \frac{-b\ Q}{LOQ_{(p,\ b)}}$$

or finally:

$$LOQ_{(p,\ b)} = \frac{(-b)(Q)}{(-p)} \qquad (5)$$

Where $LOQ_{(p,\ b)}$ is the cumulative quantity such that when the item is purased in average buy quantities Q, the relative change in expenditure in crement $(LOQ_{(p,\ b)} + Q)$ is p. This is shown on the following diagram:



Discussion: Note that equation (5) is defined by b, Q and some pre-selected fraction of decrease p. The value p should be chosen with care and will depend on the order of magnitude of the expenditure data. Equation (5) will be most valid when the production rate per time period is fairly constant.

An Example of an Application. This example concerns the production of a high volume item which is automated and involves a comparatively small direct labor effort. The data utilizes the production experiences at an Army Ammunition Plant producing a chemical propellant. This is a highly automated system which can produce several million pounds of propellant each month. Data have been obtained on these propellants over the last five years of production.

General Methodology. In order to determine if learning takes place during the manufacturing of this propellant, it was necessary to break down the available data into its elements and to evaluate each element separately.

Because this process is a highly automated process and the production volume is relatively high, the data obtained describing the behavior of the system is based on accounting information. The manufacturing costs were broken down into the following categories:

1. Direct labor costs

2. Manufacturing overhead costs

3. General and Administrative (G&A) overhead costs

Direct labor costs were used as indicators to determine if learning existed. This step is logical because the direct labor costs reflect the amount of direct labor charged to some item of production. If learning exists (as defined in this paper) then the amount of direct labor dollars charged to some unit of production should decrease as the quantity of itmes produced increases. The learning curve function will be fitted to the data and the "goodness of fit" assessed. If the product moment correlation coefficient is significant at the 5 percent level then the least squares estimate of the slope of equation (1') will be used to determine the operational level-off via equation (5).

Because the unit costs were available as dollars per 1,000 pounds of propellant, the monthly production quantities and cumulative quantities were adjusted to thousands of pounds of propellant. This was done to maintain consistency in units of measure.

The direct labor dollars were adjusted for increasing labor wages by using the actual plant average direct labor rates for calendar years 1965 to 1970 as a base. The purpose of the methodology was to relate these costs to 1970 labor rates.

These adjusted direct labor costs should now reflect the actual number of hours expended in order to produce some quantity of output (in this case propellant). The direct labor costs per unit were plotted against the cumulative quantity on a rectangular coordinate system using a CDC-6500 computer. This plot is shown in figure 2. Note that the trend of these data suggest the learning curve function described in the early portion of this paper. In order to fit this function by least squares the logarithmic form of equation (1) was used. The computer output of this statistical fit is presented in figure 3. The correlation coefficient ($r = -.89$) is significant at the 1 percent level for a sample size of 36. From a heuristic point of view, these data appear to follow, and therefore support, the existence of some learning process. The equation used to describe this process is based on experiment and the functional form is valid only in the sense of a "significant regression". What is important is the isolation of some variable (direct labor costs per unit) that can serve as some measure of learning. This variable is available through most individual accounting practices and can be utilized to assess some industrial learning phenomenon without resorting to some artificial laboratory situation.

Now that the learning curve has been determined to exist for these data, level-off considerations can be appraised. The data and regression analysis gave the following values:

(a) $b = .1537$ (slope of fitted equation)

(b) $\bar{Q} = 1980.0$ thousand lbs per month (average monthly production quantity)

(c) $SD_Q = 698$ thousand lbs per month (standard deviation of monthly production quantities)

(d) Coefficient of variation $= \frac{SD_Q}{\bar{Q}} = 35\%$. The desirability of the uniformity of "average production quantity" was discussed in the paragraphs following the derivation of equation (5).

$$Q' = \bar{Q} + 2\ (SD_Q) \qquad (6)$$

Equation (6) is a rough 95% upper limit for the monthly production quantity to be used in equation (5). This correction should be applied at the discretion of the analyst when considering the homogeneity of the production rate data.

(e) P is selected at 1%

Substituting these values into equations (6) and (5)

$$Q' = 1980 + (2)(698) = 3376 \text{ thousand lbs}$$

$$LOQ = \frac{(-.1537)(3376)}{-.01} = 52{,}000 \text{ cumulative thousand lbs of propellant}$$

The results of this analysis are shown in figure (4). The learning function exists between the points (1) and (2) and the operational level-off expenditure is between points (2) and (3).

Additional Comments: Currently, an investigation is underway to determine the outcome of fitting the function $y = ax^b + k$ to these data by least squares techniques. This equation is composed of two logical components:

ax^b = the standard learning curve

k = some hypothetical level-off time (in direct labor expenditure)

The normal equations for the least squares fit are given below:

$$\Sigma y_i = nk + a\,\Sigma x_i^{\,b}$$

$$\Sigma y_i x_i^{\,b} = k\Sigma x_i^{\,b} + a\Sigma x_i^{\,2b}$$

$$\Sigma y_i x_i^{\,b} \ln x_i = k\Sigma x_i^{\,b} \ln x_i + a\Sigma x_i^{\,2b} \ln x_i$$

These are three simultaneous non-linear equations in three unknowns (a, b and k). This "modified" improvement curve was selected as a logical starting point in trying to establish a completely objective method for defining and measuring a level-off cost (by correlation analysis). If it happens that this function is not adequate to describe the real world (i.e., k<o), then other functional forms will be attempted.

Summary. The standard industrial learning curve function is developed from the Osgood Transfer Surface. A quantitative technique for determining a learning plateau is developed. An example is presented where real-world data are subjected to the methodology outlined in the paper.

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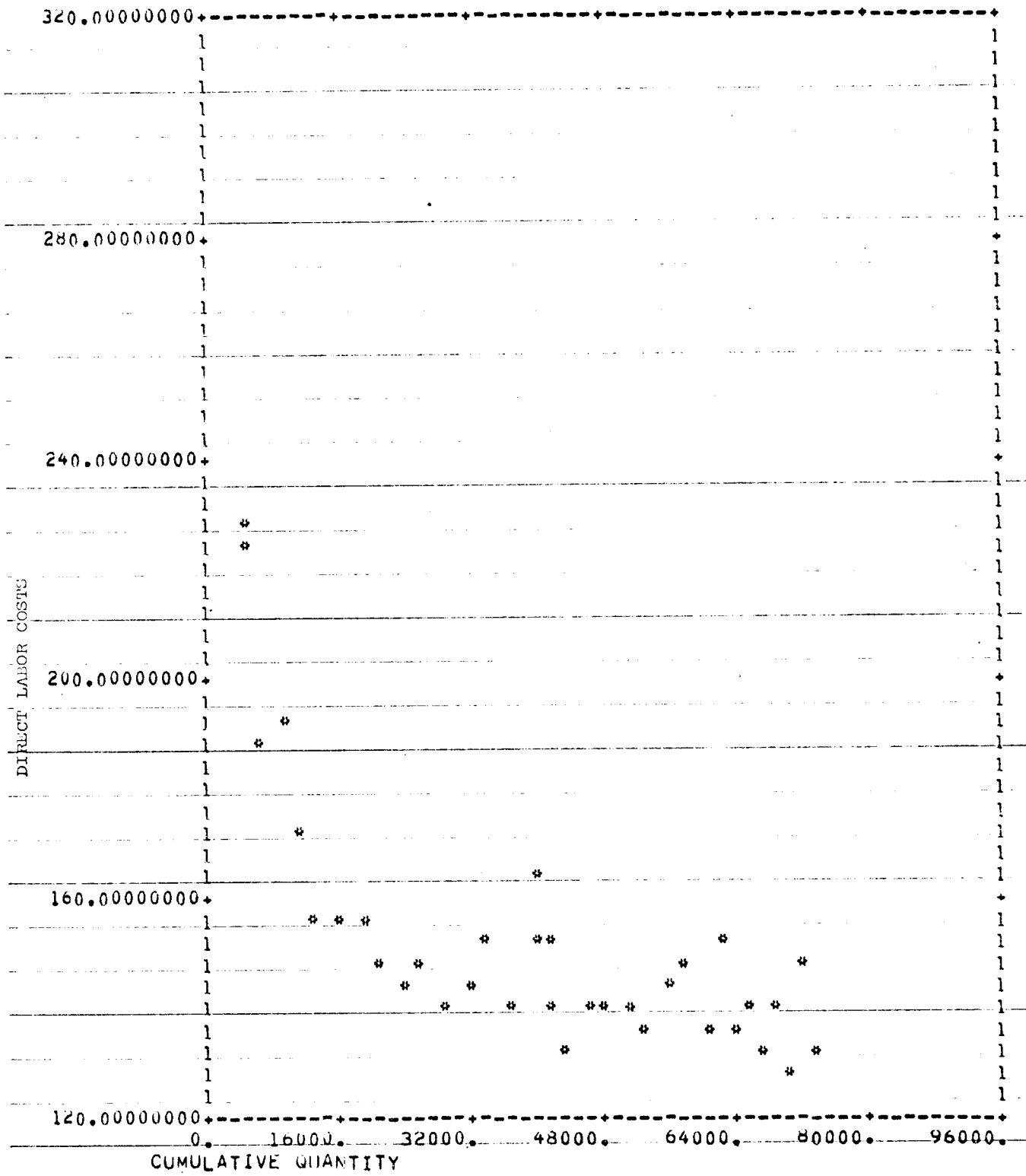
320.00000000
280.00000000
240.00000000
200.00000000
160.00000000
120.00000000
DIRECT LABOR COSTS
0.
16000.
32000.
48000.
64000.
80000.
96000.
CUMULATIVE QUANTITY

FIGURE (2)

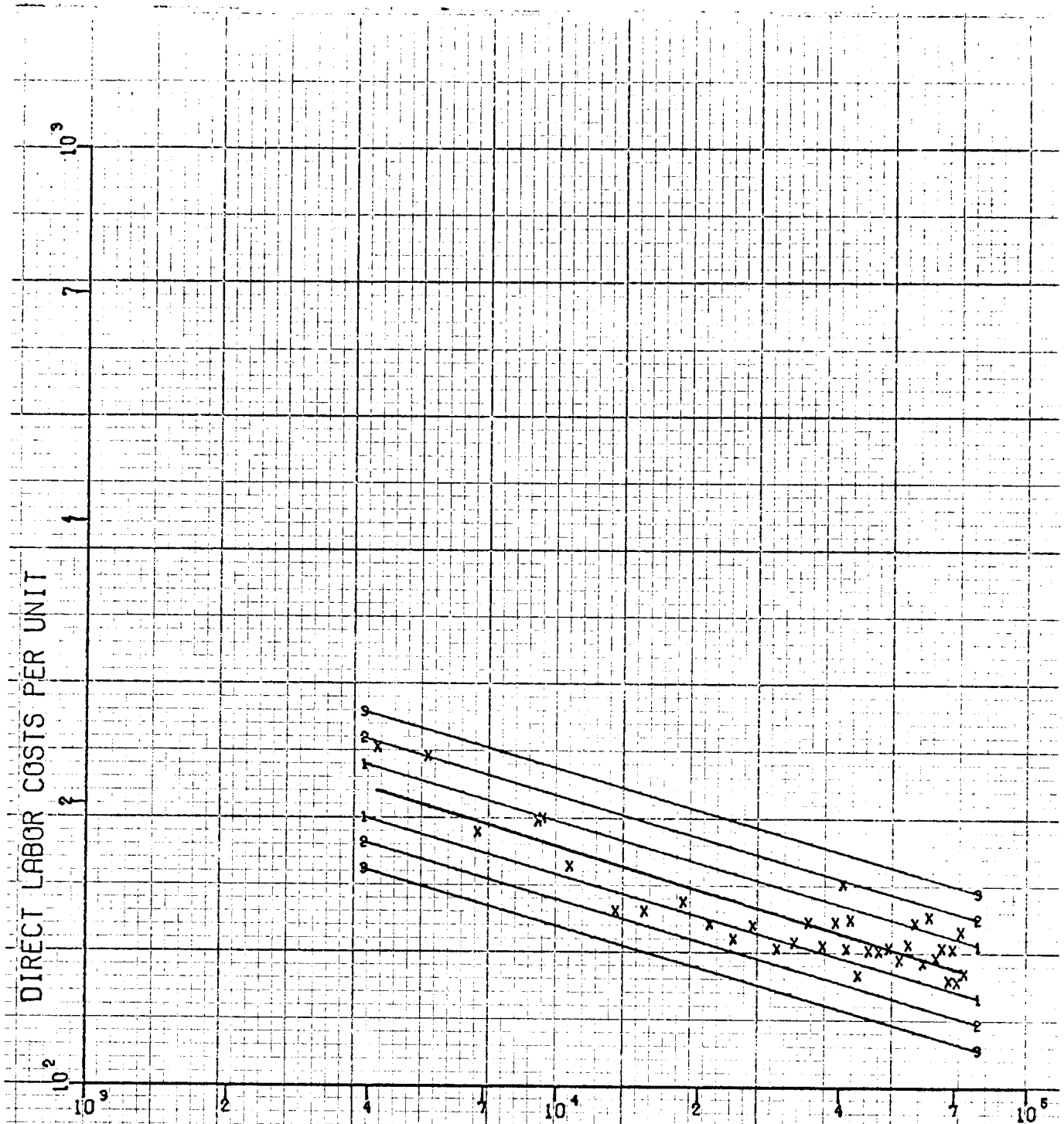
DIRECT LABOR COSTS PER UNIT
CUMULATIVE QUANTITY OF PROPELLANT
INITIAL EXPENDITURE = 743.4418
PERCENT LEARNING = 89.8968
CORR. COEFF (R) = -.8862
COEFF. OF DET. = .7854

FIGURE (3)

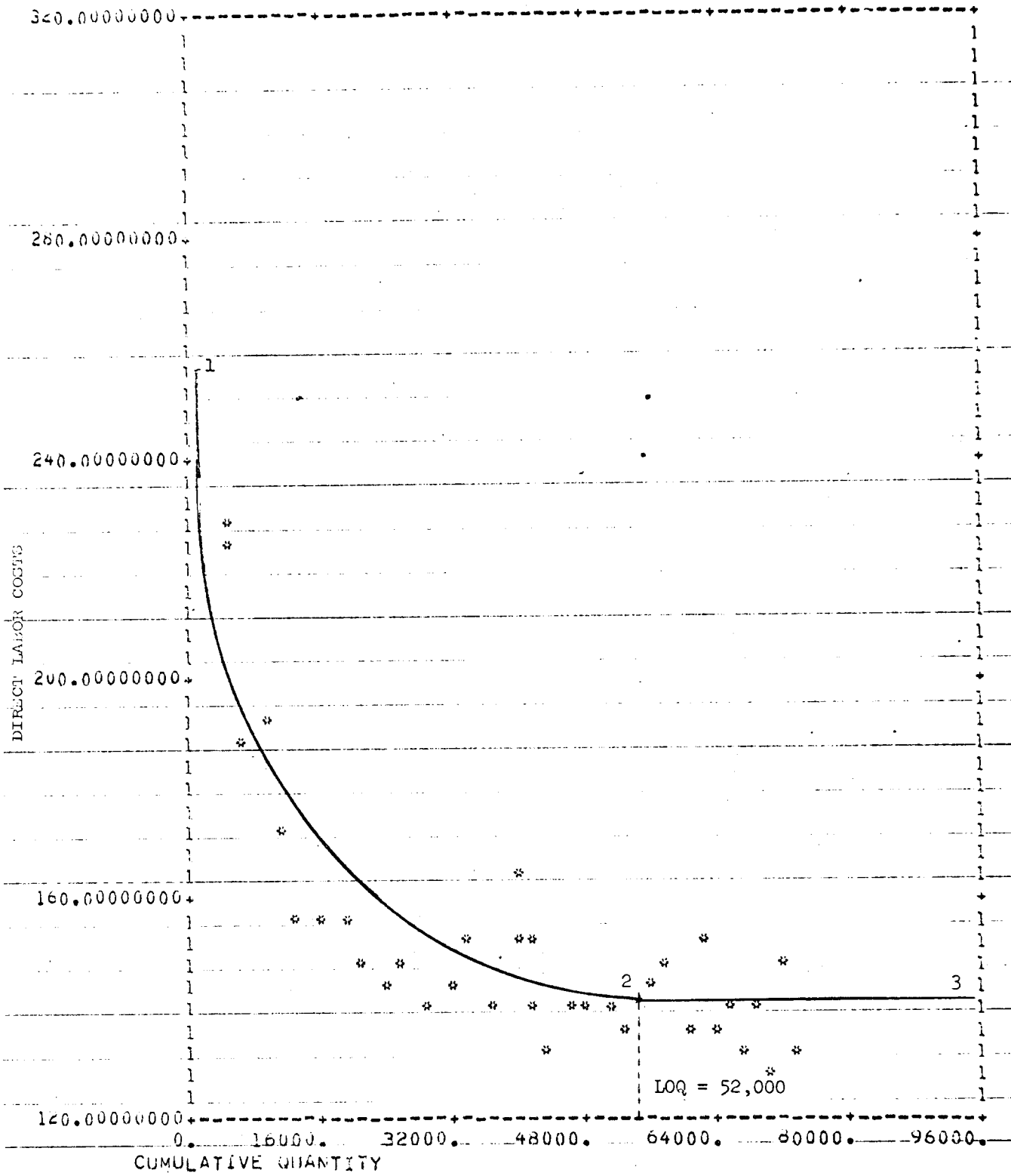
320.00000000
280.00000000
240.00000000
200.00000000
160.00000000
120.00000000
DIRECT LABOR COSTS
0.
16000.
32000.
48000.
64000.
80000.
96000.
CUMULATIVE QUANTITY
1
2
3
LOQ = 52,000

FIGURE (4)

CLASSIFICATION ANALYSIS

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ABSTRACT

Sorting things into groups is a basic intellectual task that allows people to simplify with minimal reduction in information. Classification techniques, which include both clustering and discrimination, provide step-by-step computer-based procedures for sorting things based on notions of generalized similarity and on the "class description" desired. This paper surveys and classifies all classification techniques known to the author. A conceptual framework for considering these techniques is developed. Useful techniques for interpreting and evaluating the results are given. An extensive bibliography is included.

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The remainder of the article has been reproduced photographically from the author's manuscript.

GLOSSARY OF COMMON TERMS

Algorithm: The detailed procedure or set of actions that direct a computer (or person) how to do something. The instructions you give a person that allow him to get from your house to the train station constitute an algorithm. Just as there may be several different ways to get to the train station, so there may be several different algorithms that implement the same classification technique. The power of a technique may well depend on the quality of the algorithm used in implementing it.

Category: Those objects assigned into the same group on the basis of criteria external to the classification process. All objects in the same category have the same label.

Class: Throughout this paper "class" is used to be the most general entity in classification analysis. Class will mean either cluster or category and is the entity associated with classification using either clustering or discrimination. Classification will be the general term used for processes that sort things into groups, and includes discrimination and clustering.

Class Description: The information saved by the computer program that is used to characterize the class--either cluster or category. The class description can consist of a list of the objects in the category or cluster; or some computed function of values associated with those objects--such as the average object (centroid); or a function--such as

the multivariate normal distribution--whose parameters are computed from the variable values associated with the objects in that class or cluster.

Classification: The general term used for processes that sort things into groups. Includes discrimination and clustering.

Cluster: A set of "similar" objects. Intuitively, it is a collection of objects that are similar to each other and not too similar to other objects in the data set. No external label is used to define membership in a cluster.

Clustering: (Synonyms: cluster analysis, learning without a teacher, unsupervised learning.) The finding of data-derived groups on the basis of the groups being internally similar. Does not use an externally supplied label. Terms used to describe types of clustering techniques include clumping, partitioning, and decomposition of mixtures. Numerical taxonomy applies computer-based clustering techniques primarily to biological clustering.

Criterion: An expression, usually mathematical, that expresses the user's notion of what a classification should accomplish. For example, you could use, as a criterion of the quality of a clustering, some measure of the average compactness of the clusters that are found. Number of errors is used as a criterion for categorization.

Discrimination: (Synonyms: categorization, pattern recognition, learning with a teacher, supervised learning, classification.) The finding of rules derived from the data that allow the user to assign

an unknown object to a category. Discrimination uses an externally supplied label in developing the class description.

Label: The information that associates an individual object with a specific category. Used in discrimination or to aid in interpretation in clustering.

Measure of Similarity: The expression indicating closeness (similarity or distance) between any two of the following entities (including two of the same type)--object, category description, and cluster description.

Measurement Space: The geometric space in which the relationships between the objects can be observed. A scatter plot is a representation of a two-variable measurement space.

Object: (Synonyms: pattern, vector, profile, sample, waveform, case, elements, item.) The basic entity for clustering and discrimination. An object consists of a number of measurements that characterize something. The word "pattern," which gained currency within pattern recognition, is not used here because of its multiple meanings, which include: (1) discovery of pieces of information, or (2) a vector-valued quantity on which classification procedures work, or (3) the recognition of a pattern of patterns.

Structure: The allowable relationships between entities, including specification of what is considered an entity. For example, if an object must fit into a hierarchical tree, this restricts the kind of relationships that objects can have with each other. Thus, hierarchical form is one example of structure.

Variable: (Synonyms: measurement, dimension, item, characteristic.) They characterize the object. Variables should contain the information about the object that is most relevant for classification.

1. INTRODUCTION*

Sorting things into classes--classification analysis--is an age-old problem, for which there are two major approaches: discrimination and clustering. Discrimination techniques begin with either a priori conceptual distinctions or data divided into a priori groups and proceed to develop rules by which to separate data into those a priori categories, whereas clustering techniques use a priori selection of a measure of similarity, a criterion, and a class description to find an inherent empirical structure in data--to find clusters. Thus, discrimination is the development and application of analytical rules, the parameters of which are often derived from the data, that allow a researcher to assign or sort objects into specific categories. Discrimination uses externally supplied labels associated with each member of a set of objects to aid in establishing rules for sorting things into groups. In clustering, which may be either of objects or of variables, we seek to find so-called data-derived groups based on internal similarity between the objects; the definition of similarity is left to the user, as are procedures by which sorting into similar groups is accomplished.

Workers in a variety of disciplines have taken advantage of the increased availability of the computer and more recently of man/machine interactive computer terminals to develop classification procedures and to translate them into algorithms and then into computer programs. Classification procedures have varied widely because of differing needs

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*Definitions for underlined terms appear in the Glossary.

in the various disciplines.* Because the variety is great and the literature scattered--journals that range from the Journal of Marketing to engineering journals and to texts in psychology--it is difficult for an individual to know what techniques do exist.

This paper seeks to provide the reader with a structured window into a widely scattered literature and to offer a framework within which the techniques can be organized. This framework should allow the reader to envision far more options in data analysis than are presently available in computer programs and thereby to customize his procedures to suit the idiosyncracies of his theoretical problem, on the one hand, and the nature of his data, on the other.

1.1 The ISODATA Game

Before proceeding, it should be useful to the reader to get some feeling for clustering by playing the following game. As the reader plays the game, he should try to see what things change with each step of the procedure and what information is saved from iteration to iteration. This game is essentially the ISODATA clustering algorithm (Ball and Hall, 1967b). The algorithm's simplicity allows easy exposition of a typical clustering algorithm, that is, of the detailed procedure or set of actions that the computer follows in order to sort the data into groups. The steps in the algorithm are:

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*Some references to articles describing applications of clustering to diverse disciplines are: Archaeology--Hodson (1970); Geography--Berry (1960); Economics--W. D. Fisher (1969); Electrical Engineering--Alens (1967); Information Retrieval--Dale and Dale (1965); Market Analysis--Green et al. (1968); Medicine--Stark et al. (1962); Numerical Taxonomy--Basford et al. (1968); Psychology and Sociology--Katz et al. (1968); Statistics--Dalenius and Gurney (1951); and Weather Prediction--Endlich (1970). The last describes an unusual application of a clustering algorithm.

(1) Seat a group of persons in a room so that people are divided into subgroups.

(2) Arbitrarily select two people to be initial cluster reference points.

(3) Everyone in the room is to point to the person who is the closest cluster reference point. All of those pointing to the same cluster reference point are considered to be in the same cluster.

(4) Find the average location point of those associated with each location.

(5) Have everyone now point to the newly found closest average location point. This defines new cluster membership. From the new cluster memberships, new cluster averages can be found. These imply new assignments to clusters.

(6) Repeat this process until no person is reassigned.

(7) At any time the number of clusters can be increased either by breaking an existing cluster into two clusters or by selecting additional persons to serve as cluster reference points.

(8) The number of clusters can be reduced by combining two adjacent clusters and then determining a new cluster average point for the combined cluster.

1.2 The Rosen-Hall Mode-Seeking Discrimination Game

A similar game can be created using a discrimination algorithm (Rosen and Hall, 1966). Briefly, the game goes as follows:

(1) Assign all persons sitting in the room to one of three categories. (In the case of discrimination, each person--persons are the objects to be sorted--must have been assigned a category label.) Each of these three categories should have several modes--several distinct groups of persons. The persons in each mode should be from predominantly one category.

(2) Arbitrarily select one person within each class as the mode-seeking reference point for that class for the first partitioning.

(3) Everyone points toward the closest mode-seeking reference point, regardless of the category associated with that modal point.

(4) Everyone not pointing to a person in his own category should stand up.

(5) Select additional persons to represent each category from among those who are now standing up. At the same time, improve the category descriptions for those sitting down by finding the average person within each class. (If two or more averages are being used within any one class, then each person affects the modification of only his closest within-class average.)

(6) Evaluate the quality of sorting on the basis of the existing modes. If the error rate, that is, the misassignment of people based on closest mode, is still unacceptably high, then return to Step Two. If the error rate is acceptable, considering the number of

cluster centers that have been generated and the presumed quality of the data, then terminate the process.

These games emphasize that iterative methods that seek to improve a sorting can find sortings that are intuitively satisfying. In what follows, we discuss the motivation and the context for classification techniques, examine in detail the elements of classification techniques, construct a framework for organizing classification techniques, and consider ways to interpret the output from classification techniques.

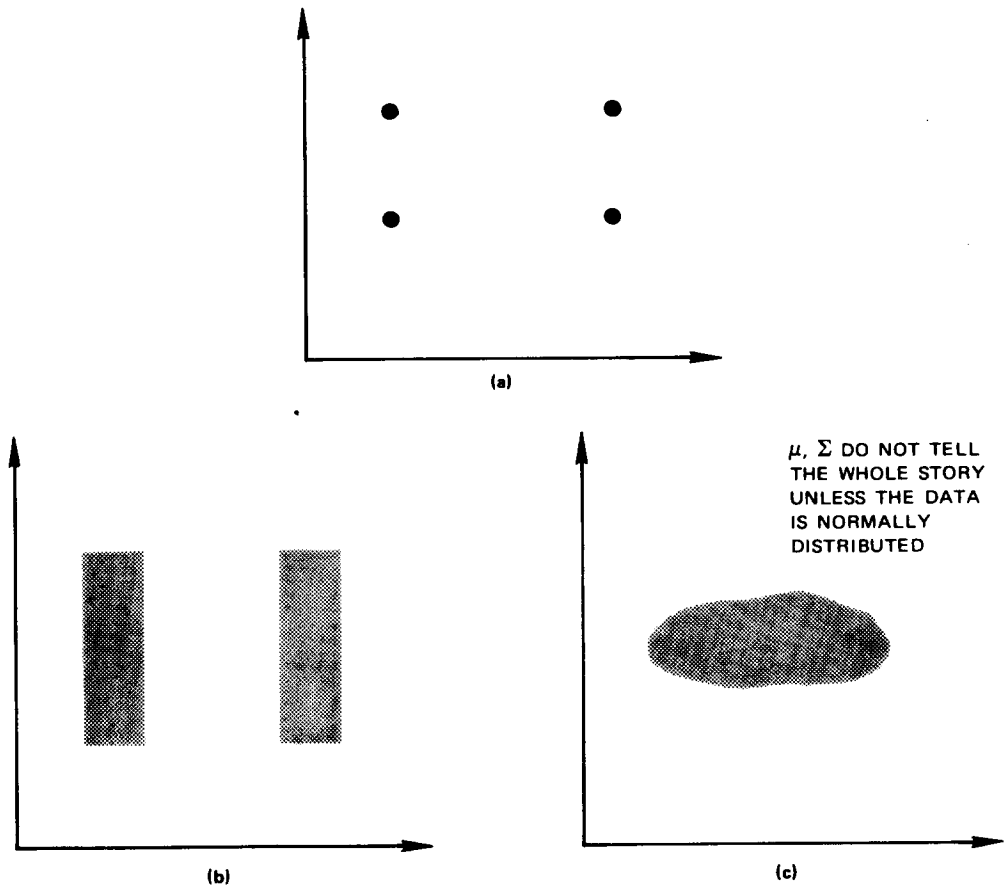
1.3 Similar Statistics from Dissimilar Data

Part of the motivation for multivariate classification techniques arises from the desire to reduce the summarization that can mask dissimilarities in multivariate data. Consider, for example, Figure 1, in which three dissimilar data sets yield identical mean values and identical covariance matrices.

From this example, it should be clear that the mean and the covariance matrix do not tell the whole story unless the data are normally distributed. Techniques that use only values of a mean vector and a covariance matrix, and do not use original values of data again after the mean and covariance have been calculated, cannot distinguish between these three data sets. Techniques that use additional information do exist and do indicate the difference, as we shall see below.

A second consideration, more specific to clustering, concerns what a user means by a natural cluster. If we examine Figure 2, we see that something as arbitrary as the scaling of individual variables can affect how objects are placed into a cluster.

In Figure 2(a), we might be inclined on the basis of physical inspection to consider Clusters 1, 2, 3, and 4 as distinct. By rescaling



Three dissimilar sets of data all with the same mean and the same covariance matrix
(a) Data concentrated at four points
(b) Data spread uniformly in two rectangular regions
(c) Data distributed in accordance with a bivariate normal distribution

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FIGURE 1 DISSIMILAR DATA WITH IDENTICAL MEANS AND COVARIANCE MATRICIES

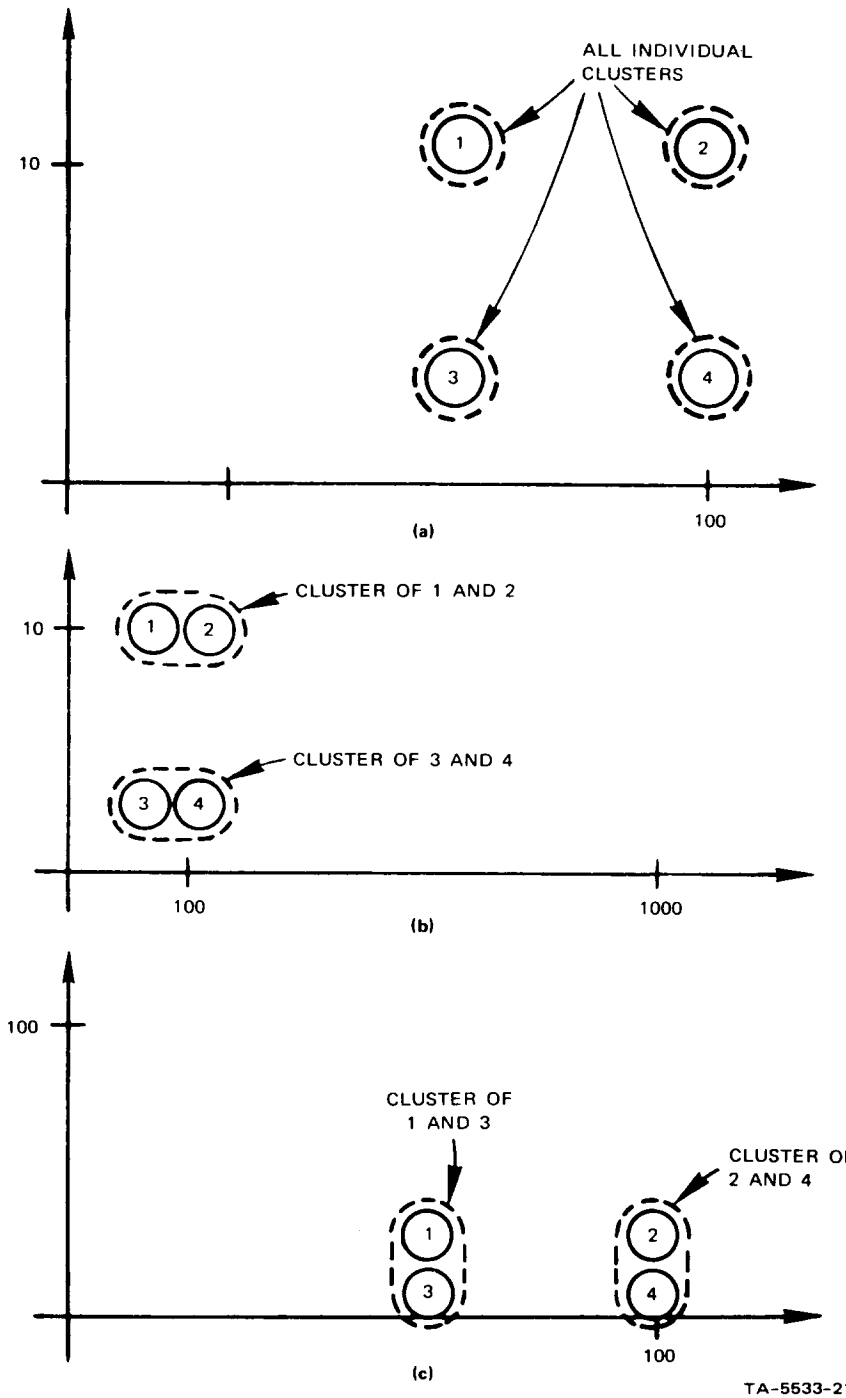

FIGURE 2 THE EFFECT OF SCALING ON WHICH CLUSTERS LOOK SIMILAR

the X axis in Figure 2(b), it is possible to believe that Clusters 1 and 2 are, in fact, a single supercluster, as are Clusters 3 and 4. It may be somewhat disconcerting to note, however, that a scaling of the Y axis [Figure 2(c)] could reduce the four clusters to two superclusters--but this time with Clusters 1 and 3 placed together, and Clusters 2 and 4 together.

Thus we see that the context for classification techniques must be used in interpreting the results. Certain key questions must be answered in order to establish that context.

2. KEY QUESTIONS

A user will find it necessary to answer certain questions in sorting objects or variables into groups. How these questions are answered by a user should provide him with a guide for selecting (or developing) appropriate classification techniques. Some basic theoretical questions are as follows:

(1) What are the goals of the user?

(2) What assumptions are made about the underlying structure of the data?

(3) Is the user willing to assume that his knowledge of class membership is sufficient to guide the grouping procedure? In short, does he have examples of well-defined groups on the basis of which he wishes to develop a discrimination scheme, or does he have a relatively undifferentiated set of objects whose underlying structure he wishes to explore by clustering on the basis of interobject similarity?

(4) Does the user primarily wish to group variables or to group objects?

(5) How do the user's particular research question and specific paradigm affect answers to all of these questions?

A second set of questions concerns the nature of the variables and objects:

(6) How are the variables and objects determined to be worthy of study in the first place?

(7) How are the variables scaled and counted?

(8) Is redundancy, such as high intercorrelation between variables, to be removed?

(9) How are individual variables (or objects) combined into a single number that can be used to measure similarity between objects (or variables), or to measure the similarity of an object (or variable) to a class?

A third set of questions to consider relates to the availability of computer capabilities:

(10) What techniques and existing programs contain the desired ingredients? How well does an implemented program accomplish its theoretical objectives?

(11) Can classes of techniques be identified? What are the characteristics of each class, and what is focused on in the definition of that class? What is the characteristic algorithm in each class?

(12) What quantity of monetary and time resources is a user willing to devote to the analysis? How many repetitions of the analysis cycle are acceptable? How cut-and-dried need the results of the analysis be?

(13) What other techniques does the user plan to use in conjunction with techniques for sorting into groups? How should results from various techniques be integrated to provide a coherent overall view of the data?

(14) With what language--for example, statistics, probability, or graph theory--does the user feel most comfortable?

(15) What aids exist for interpretation and evaluation of results?

(16) How would a user prefer results of sorting be presented to him?

(17) What criteria should be used in evaluating results?

We will spend the remainder of this report dealing with the issues raised by the seventeen questions presented above.

3. GOALS

It is important to realize that goals of various users of clustering and discrimination techniques are frequently dissimilar. Once this is realized, it is easier to see why such a variety of clustering and discrimination techniques exist and also to avoid somewhat fruitless arguments as to which technique is the most ideal in some global sense. Some salient goals are the following:

(1) finding a true typology

(2) model fitting

(3) prediction based on groups

(4) hypothesis testing

(5) data exploration

(6) fishing in data to find useful hypotheses

(7) data reduction.

Some of these goals may work at cross purposes. For example, procedures for conducting a fishing expedition will clearly be more relaxed than those used to fit data to well-articulated models.

4. AN IMPORTANT DIGRESSION

It is important for the user of classification techniques to consider quite consciously the interaction between technique and discipline context. In Figure 3 we see some suggestion of the interaction between the theories and methods.

On the one hand, we have a particular question that we want to learn more about plus information regarding the probable structure of relevant data. This question implies a selection and preprocessing of variables and objects. Underlying the question is a paradigm, which aids us both in interpreting the data and in predicting probable structures of the data. On the other hand, we have techniques, probably based on mathematical assumptions. Characteristics of the techniques also influence the selection and preprocessing of variables; the user must define explicitly what he means intuitively by similarity between objects. His explicit definition must then become embodied in a computer algorithm, which has implications for the structure of the data. This algorithm, in combination with auxiliary techniques, leads to an interpretation of the structure of the data, which may in turn entail either a modification of the theoretical paradigm or a reconsideration of our empirical analysis.

In sum, implicit and explicit judgments regarding scaling, criteria chosen, selection of objects, and selection of variables for inclusion in a body of data affect the outcome of a classification analysis. A slogan that captures some aspects of the situation is that "explicitness is not objectivity"; simply because a researcher chooses one method of classification does not mean that he has somehow found Truth. He has

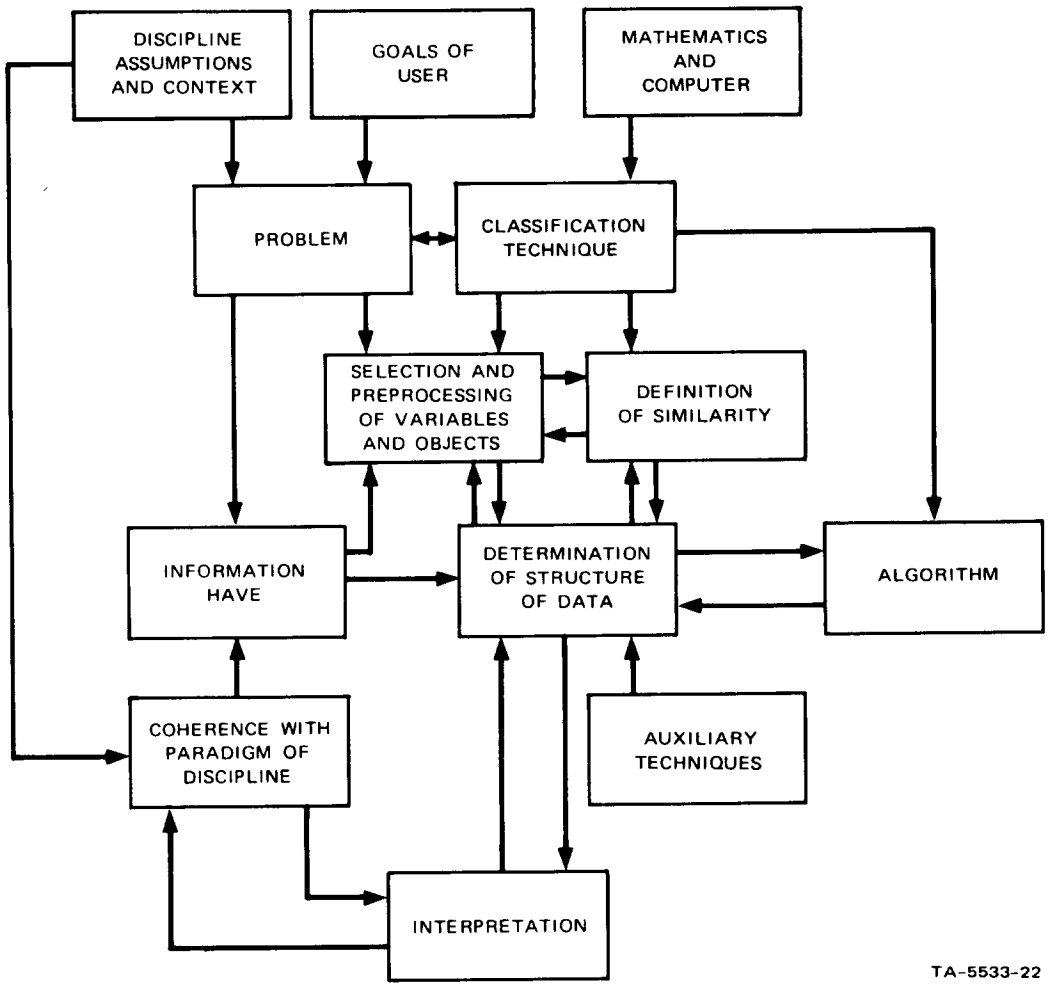

FIGURE 3 THE INTERACTION BETWEEN DISCIPLINE AND METHODOLOGY

made it easier for others to see why he has grouped things together, and therein lies the value of explicitness, but metaphysical questions about the nature of reality cannot be settled by making a priori methodological choices before analyzing data. The question of selecting an appropriate technique for a particular set of data will remain problematic even when results are obtained through classificatory analysis, but they can never be resolved through methodological decisions made out of context. The resolution of the questions still requires capable people with experience and overall understanding in the relevant fields of inquiry.

5. PROCESSES FOR SORTING THINGS INTO GROUPS

Let us now return to processes for sorting things into groups. Consider Figure 4, in which information is shown as flowing from the environment, a set of measurements is taken, and a classification is produced.

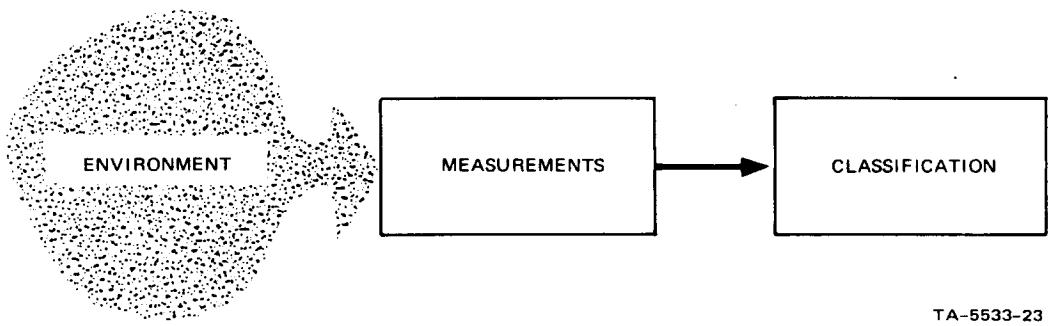

FIGURE 4 THE CONTEXT OF CLASSIFICATION

This diagram--with its emphasis on the choice of environment and on data preprocessing--is much used in pattern recognition with its primary focus on discrimination, where success usually has depended on skillful choice of the measurements used to discriminate between classes. [See Nagy (1968) for an excellent discussion on pattern recognition.]

Expansion of the flow chart of Figure 4 into that shown in Figure 5 suggests that a researcher's decisions on many factors are important to the classification process.

A researcher chooses a portion of the environment and a set of instruments to handle his data. In the social sciences the instruments

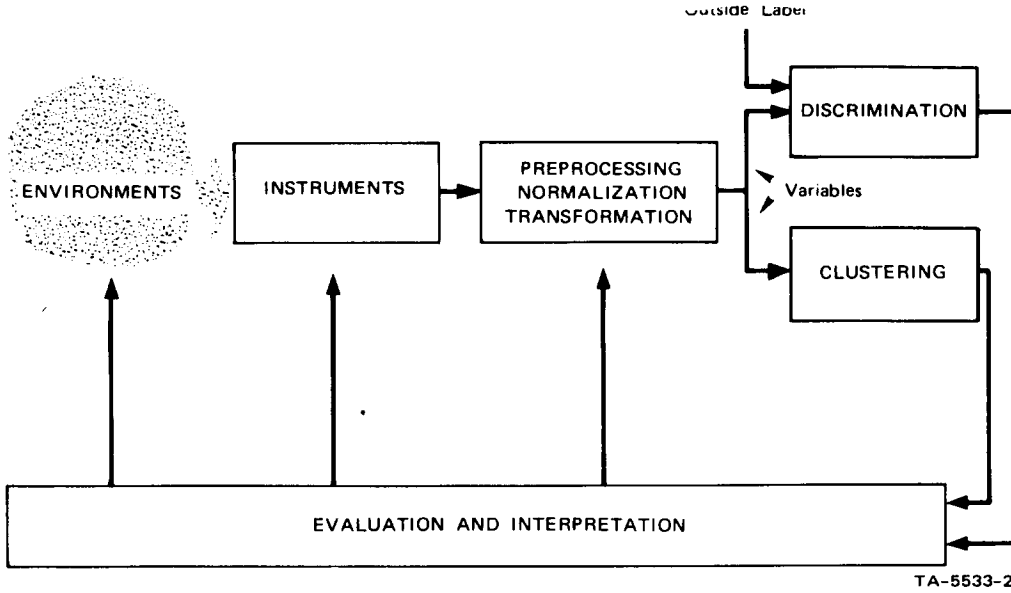

FIGURE 5 DETAILED FLOW CHART OF THE CONTEXT OF THE CLASSIFICATION PROCESS

might be, for example, census statistics or nominal scale responses to questionnaires. Data from such instruments must then be preprocessed, normalized, preformatted, and somehow turned into numbers associated with particular variables* such that one can perform either a deductive discrimination or an inductive clustering of the data. In both discrimination and clustering, evaluation and interpretation can result in modifications in any aspect of the earlier stages. We may choose a slightly different portion of the environment to examine, for instance; we may improve our instruments based on our initial experiments; we may modify our preprocessing in such a way as to remove variations irrelevant to our classificatory interests; or we may redefine our variables so that our classification procedures are simpler.

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*Multidimensional scaling, which uses proximities (Shepard, 1962) and some classification procedures--notably the clumping procedures described in Section 12.2--can use an implicit definition of similarity. Underlying variables need not be defined.

6. DATA STRUCTURE AS IT AFFECTS CHOICE OF TECHNIQUE

Assumptions about the structure of data should affect our choice of a technique. The assumptions can be usefully characterized in terms of two dimensions. The first dimension considers the amount of overlap we assume to exist between groups that we are to sort. The second considers the variety of the form and shape of data in the group. Both of these dimensions are indicated schematically in Figure 6.

In Figure 6(a) we have two groups of nearly identical shapes that are widely separated from each other. In Figure 6(f) there are two groups that differ considerably in shape but whose means are identical.

The importance of the structure of the data relates to assumptions we are willing or are forced to make in order to elicit information from the data. For example, in Figure 6(c) we can assume that the number of groups is two and that they have equal covariance matrices. With these two assumptions, we may be able to obtain satisfactory estimates of characteristics for both groups. If, however, our assumptions are incorrect, we may be imposing a structure on the data more than finding one in it. The sensitivity of our results to our assumptions obviously needs to be checked. Due to the existing state of the art of classification analysis, it is not always clear how these assumptions can be tested.

We have suggested that externally supplied labels for objects vary in their level of validity. It may be helpful to use both clustering and discrimination procedures on a set of labeled data and thus, in some sense, to cross check the validity of one's labels. Consider these examples: The sorting of characters from a particular typewriter into letters of an alphabet is an instance in which we could supply almost

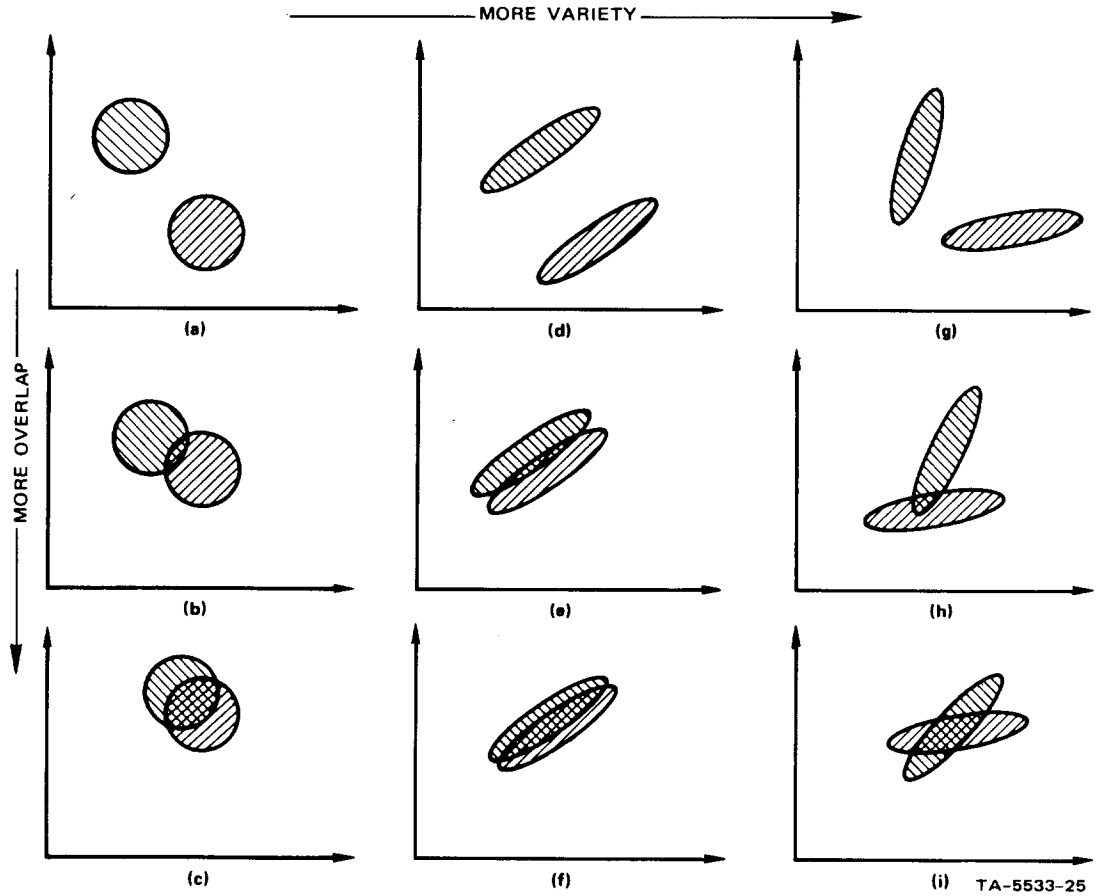

FIGURE 6 THE STRUCTURE OF DATA

completely valid labels. The sorting of dysfunctional persons into categories as in the field of psychiatry may be an example where existing labeling should be questioned. Using labels provides us with a way of determining the power of certain variables for sorting objects into assigned groups. On the other hand, clustering without labels can give an indication of the extent to which categories or classes overlap within a particular set of variables. In both cases the information gained may

suggest a modification and reevaluation of classes, of variables, of procedures, and of objects.

Assumptions about the structure of the data influence our choice of algorithm and our definition of similarity. Thus our confidence in the quality of available external labels that could be used to divide objects into groups affects whether we choose to engage in discrimination or clustering.

In short, there is considerable similarity between clustering procedures and discrimination procedures.

7. THE SELECTION OF VARIABLES AND OBJECTS

A researcher must choose variables and objects before using a particular classification technique. If he is performing discrimination, he will wish to find variables that enhance differences between classes while retaining some amount of within-class similarity. If he is clustering, he will seek measurements that highlight intraclass similarity while at the same time taking into account differences found between groups.

Once variables have been selected, they can be scaled, coded, and transformed in a variety of ways. We have already discussed how scaling affects particular techniques. It is less clear how each variable should be weighted and whether natural or transformed variables are more appropriate.* At this point it may be important to note again that a paradigm may answer some of these questions in an a priori manner, and often there is no alternative to complete dependence on a paradigm. One can, however, examine the sensitivity of groupings to alternative transformations of the data, variations in coding, and different scalings. If groupings tend to be relatively invariant under these changes, then one might conclude that one's empirically defined groupings are not merely artifacts of analytical assumptions and a priori choices.

Consider the process by which one gets numbers out of an environment. In Figure 7 we show a leaf and indicate ways to measure its width

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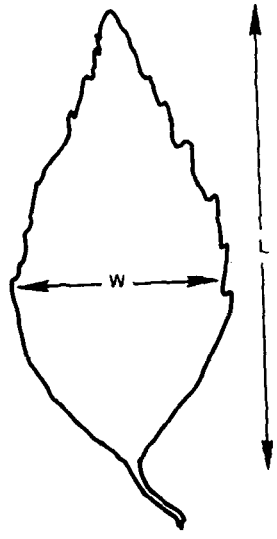
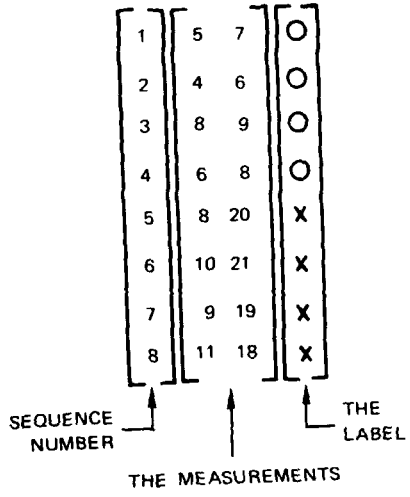
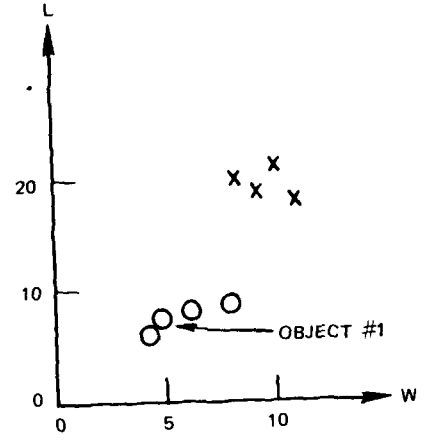
*There has raged a controversy in numerical taxonomy as to the weighting of variables for some years. Probably this weighting has occurred subjectively in all classifications, but the use of the computer forces the researcher to make an explicit decision as to the weighting.



(a) A LEAF



(b) MANY LEAVES



(c) THE PLOT

FIGURE 7 THE PROCESS OF GETTING NUMBERS

and length. There are many ways of defining the width and length of a leaf, and experience within the discipline of botany should suggest measurements that are less apt to introduce irrelevant variation.

Measurements obtained can be organized into a matrix such as the one shown in Figure 7(b), which also includes a column containing the label for each leaf. The label has, in all probability, been supplied by a botanist, who says, "These are the same kind of leaves, and they differ from another group over here." We can plot the measurements in a measurement space, as shown in Figure 7(c). When this is done, it appears that we have two distinct groups. From plots of numbers we form concepts, such as dimension and closeness. [A recent paper (Zahn, 1969) relates differing Gestalt perceptions to clustering procedures.]

8. MEASURES OF SIMILARITY

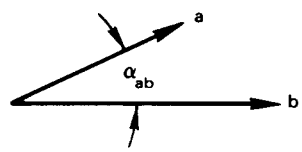
In order to use either clustering or discrimination, it is necessary to select a criterion of similarity (or dissimilarity). The measure of similarity indicates closeness (similarity or distance) between any two of the following entities (including two of the same type): object, category description, and cluster description. If we wish to compare two entities described by numerical values for a number of variables, we must combine differences for each variable in this set of variables into a single number. Although variables can be extremely dissimilar analytically, at the empirical level the use of a similarity measure assumes numerical comparability between variables since it combines effects of individual variables into a single number--the number giving the similarity between two objects.

This assumption of numerical comparability allows simple clustering processes that group objects by overall similarity. A single measure of similarity can be used to determine the clustering. After clustering, the contribution of individual variables can be determined by decomposing, say the sum of squared error curve, into the contributions from each of the individual variables (cf. Section 13.2). Thus, we can first use the simplicity of a single number to measure similarity and then decompose each cluster into distributions of values for each variable, retaining distinctions between variables while still being able to group objects on the basis of overall similarity.

It is not necessary to combine all variables using a similarity measure. The automatic interaction detector (Sonquist and Morgan, 1964) orders variables and uses them one at a time in partitioning a data set,

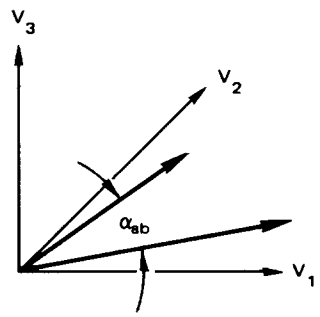
eliminating the need for a measure of similarity to group contributions of the many variables. Hartigan (1969) considers ways in which the identity of an individual variable can be retained, at least part way through the clustering. This procedure looks only at distinctions based on a single variable at each phase of the clustering process. (Single-variable partitioning has very old antecedents and underlies the sequence of one-variable classifications used in taxonomic keys.)

To give the reader some feeling for a measure of similarity, we show in Figure 8 the measure of similarity known as correlation as an



$$S_{ab} = \cos_{ab} = \frac{a_1 \cdot b_1 + a_2 \cdot b_2}{\left[(a_1^2 + a_2^2)(b_1^2 + b_2^2)\right]^{1/2}}$$

(a) TWO DIMENSIONS



$$S_{ab} = \cos_{ab} = \frac{a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3}{\left[(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)\right]^{1/2}}$$

(b) THE GENERALIZATION TO 3 DIMENSIONS

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FIGURE 8 CORRELATION AS A MEASURE OF SIMILARITY

angular measure between two vectors with respect to a specific origin position. Note that the two-dimensional definition of correlation can be generalized directly to a measure of similarity using more than two

variables, as shown in the figure. Similar figures can be drawn for other measures of similarity.

We can identify a number of different kinds of measures of similarities. Examples of these are given in Table I.

(1) Association--Used primarily for information retrieval and biological taxonomy (cf. Kuhns, 1964, and Goodman and Kruskal, 1954). Some forms of measures of association are also measures of correlation for binary variables.

(2) Correlation--Primarily a function of the angle between a pair of object vectors, it can be normalized or left unnormalized.* In either case, a correlation is sensitive to the position of the origin, and values of correlation change if the position of the origin changes. Correlation is most useful as a measure of similarity when shape--the pattern of ratios of the various variables--is to be the prime determinant of similarity. For example, a person interested in the shape of human bodies might wish only to consider relative proportions of a group of people on whom he had collected measurements. (See Cattell, 1949 for a related measure of similarity.)

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*Normalized correlation depends only on the angle between the object vectors. Normalized correlation is most useful as a measure of similarity when the magnitude of the object does not affect the clustering or the discrimination. Unnormalized correlation weights the angle between two objects by the magnitude of the objects with respect to the present origin. Unnormalized correlation seems an unhappy compromise between normalized correlation and distance as a measure of similarity.

Table I

MEASURES OF SIMILARITY

Association	$S_{xy} = \dfrac{n_{xy}}{n_{xx} + n_{yy} - n_{xy}}$ where n_{ij} is the number of 1's that occur in x and y in the same variable.
Correlation	$S_{xy} = \dfrac{x_1 y_1 + \ldots + x_v y_v}{\sqrt{\left(x_1^2 + \ldots + x_v^2\right)\left(y_1^2 + \ldots + y_v^2\right)}}$ where v is the number of variables.
Distances:	
Absolute	$D_{xy} = \lvert x_1 - y_1 \rvert + \ldots + \lvert x_v - y_v \rvert$
Euclidean	$D_{xy} = \sqrt{\left(x_1 - y_1\right)^2 + \ldots + \left(x_v - y_v\right)^2}$
Weighted Euclidean	$D_{xy} = \sqrt{w_1\left(x_1 - y_1\right)^2 + \ldots + w_v\left(x_v - y_v\right)^2}$ where w_i are variable weights.
Mahalanobis	$D_{xy} = \sum_{i=1}^{v} \sum_{j=1}^{v} w_{ij}\left(x_i - y_i\right)\left(x_j - y_j\right)$ where the w_{ij} depend on the scatter matrix. (See Appendix A.)
Probabilistic (use +1 if $x_j = y_j$ use -1 if $x_j \neq y_j$)	$\sum_{j=1}^{v} (\pm 1)\log\left[1/\text{prob}_j\left(x_j, y_j\right)\right]$
Functional	$S_{ij} = \dfrac{1}{\left(1 + D_{xy}\right)}$

(3) Distances--This measure, whose values run the reverse of similarity, may be determined in at least five ways:

Absolute "City Block" Distance. Less sensitive to a single variable's having a large difference than are the following, higher order measures of distance.

Euclidean Distance. Sensitive to the scaling of the variables that make up the object vector but insensitive to the location of the origin. Geometric distances are used widely but can yield misleading results unless normalizations are performed properly.

Mahalanobis Distance. Weightings are data-derived in Mahalanobis distance by using a within-group covariance or correlation matrix (Mahalanobis, 1936). In Friedman and Rubin (1967), the within-group covariance matrix changes as the assignment of objects to classes changes. The choice of the within-group covariance matrix affects the value of Mahalanobis distance substantially. This matrix can be formed by considering the entire data set around the overall average, or by averaging together matrices around each class center (i.e., after subtracting the mean of the class), or by having each class have a different matrix.

General Weighted Distance. Weightings can be derived either from an a priori evaluation of

the importance of variables or from some other procedure, such as discriminant analysis.

Minkowski Distance. This measure of similarity includes the absolute and Euclidean forms of distance by setting the power of the expression (x-y) appropriately. For example, if the power is 2, then Minkowski distance specializes to Euclidean distance.

(4) Probabilistic Measures--Useful when it is appropriate to use the population statistics to modify weightings of variables. When the weighting is determined by the population statistics, however, these measures become quite sensitive to the choice of the population. The weighting depends on the value of the variable, as well as the particular variable used as in weighted distance; i.e., the weighting of the significance of the similarity will, in general, be different if the two objects being compared have values 1 and 1, as opposed to having values 0 and 0. While past authors have used weightings that were functions of population statistics, the user could develop his own process for assigning weightings.

(5) Functional Measures of Similarity--The value of the measure is a function of d, the distance from the other object or class.

By way of comparing these measures of similarity and to give the reader some indication of the effects his choice of similarity measure has, we indicate in Figure 9, by using contours of isosimilarity, the

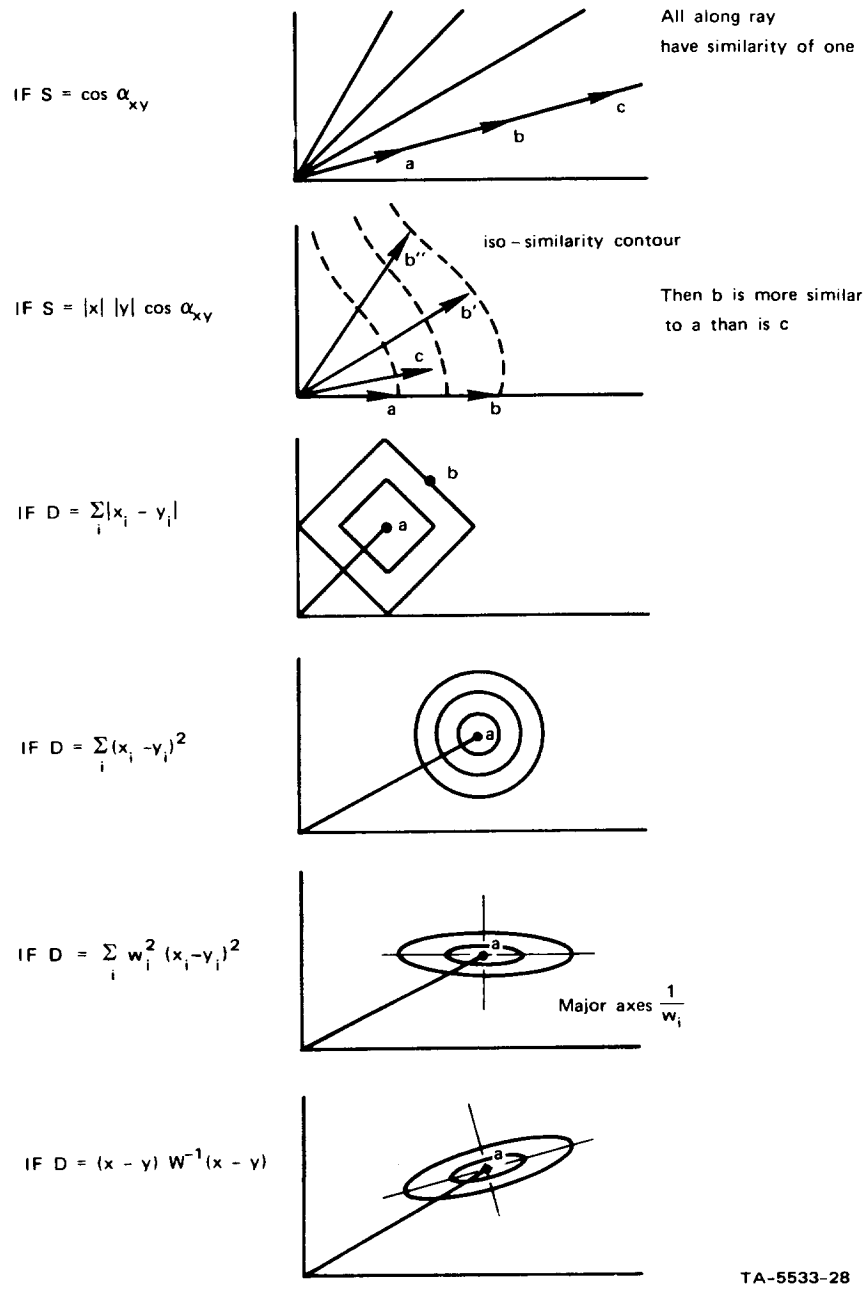


FIGURE 9 ISO-SIMILARITY CONTOURS FOR VARIOUS MEASURES OF SIMILARITY WITH RESPECT TO VECTOR a

two-dimensional objects that would be considered similar using each of these measures. In other words, any two objects lying on the same isosimilarity contour would be equally similar to the basic object shown as vector a.

Note that if a functional measure of similarity is a monotonic function of distance, then the form of isosimilarity contours is the same as that of the distance of which the measure is a function. The spacing of the contours will, in general, be different. The difference in spacing will have an additional effect if the number calculated for similarity is used for calculating intragroup similarity or for weighting objects in calculation of the class average. For example, decomposition techniques (Section 12.4) tend to use $p(x)$, the probability of membership in that class, as the weighting factor rather than weighting all objects equally.

9. CLASS DESCRIPTIONS AND THE SORTING PROCESS

As we examine clustering and discrimination techniques, it will become increasingly apparent that measures of similarity can be constructed not only between pairs of objects, but also from an individual object to an entire class--to a cluster or to a category. It becomes important then to consider class descriptions.

The information saved by the computer program and used to characterize the class can be called the class description--describing either cluster or category. The class description can consist of a list of the objects in the category or cluster; or some computed function of values associated with those objects--such as the average object (centroid); or a function--such as the multivariate normal distribution--whose parameters are computed from the variable values associated with the objects in that class or cluster.

Three primary forms of class description exist:

(1) We can describe a class by listing the identification numbers of the individual object's in that class. This class description is largely used in single-linkage clumping or graph theoretic techniques.

(2) We can extract from a class some descriptive states such as an average, and we can ascertain similarity to that average. The class average can be computed by weighting the objects in at least two ways--by 1/N, the most common weighting, or by p(x), the probability of that object belonging to that class.

(3) A third alternative is to describe a class by a functional form; for example, we can use a normal distribution with a standardized mean and covariance matrix. (When the class is described by a functional form, we can use the value of the function at the location of an object as a measure of the similarity of that object to the class.)

Classification may be conceived of as an iterative process in which information gained from a prior sorting aids development of a class description, perhaps also to modify the measure of similarity. Consider, for example, the process shown in Figure 10.

Let us start, assuming that we have preprocessed our data, decided on a measure of similarity and on a procedure for sorting--either clustering or discrimination. If we choose to do clustering, then we must select a criterion. The criterion may be implicit in the technique. For example, ISODATA tends to minimize sum of squared distances, but the criterion is not optimized explicitly. If we choose to do discrimination, then labels are supplied for the data on a theoretical or a pragmatic basis. In both cases the data are used to develop a class or clustered description. This class description governs the next sorting of the data.

The process repeats iteratively: Each time the sorting uses the most recent class description, which in turn tends to reassign objects, which then leads to a new category or cluster description. These new descriptions govern the subsequent sorting. In addition to changes in the class description, the parameters of the measure of similarity may change, as in Friedman and Rubin (1967), where the intracluster covariance matrix is recomputed as objects are reassigned to new clusters.

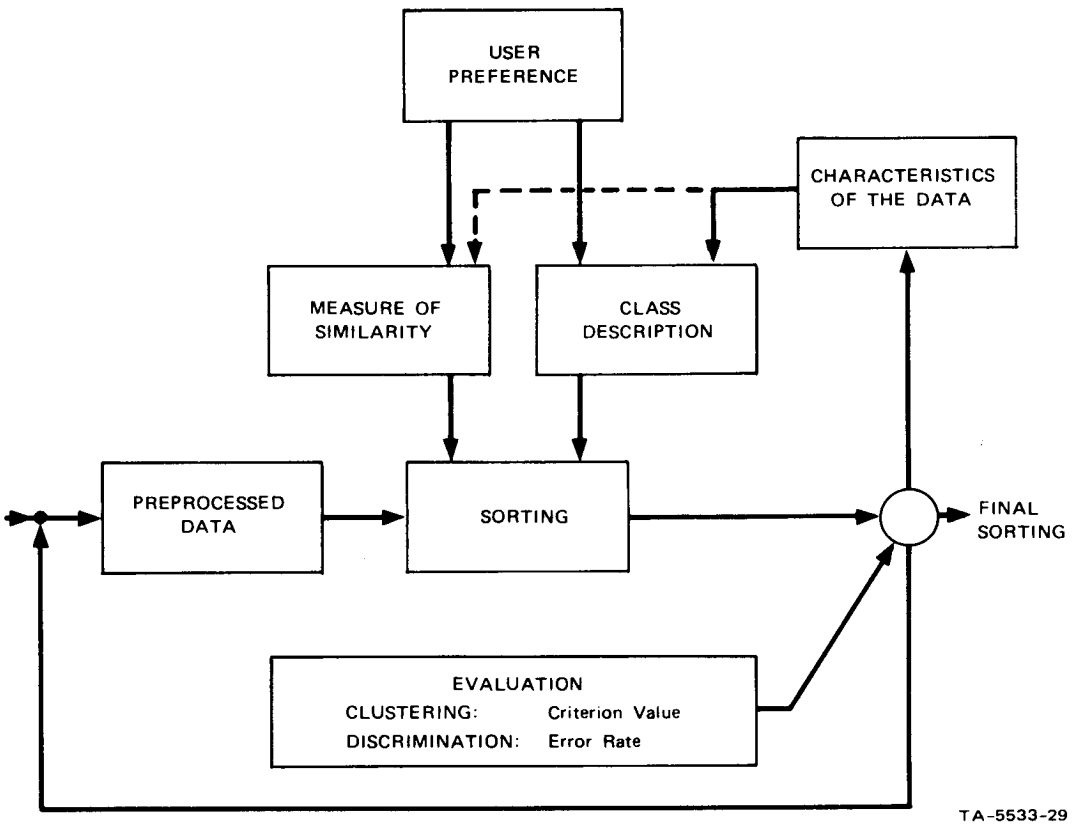


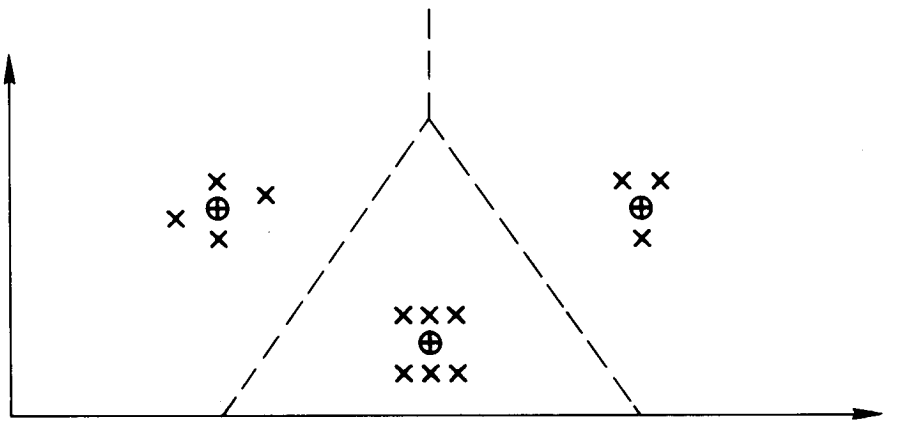
FIGURE 10 THE PROCESS OF CLASSIFICATION

At some stage, which may be during the computational process itself, the clustering or discrimination is evaluated. Clustering tends to be evaluated on the basis of criteria related to within-class similarity relative to between-class dissimilarity; discrimination usually is appraised by looking at the number of errors resulting from the application of an assignment rule.

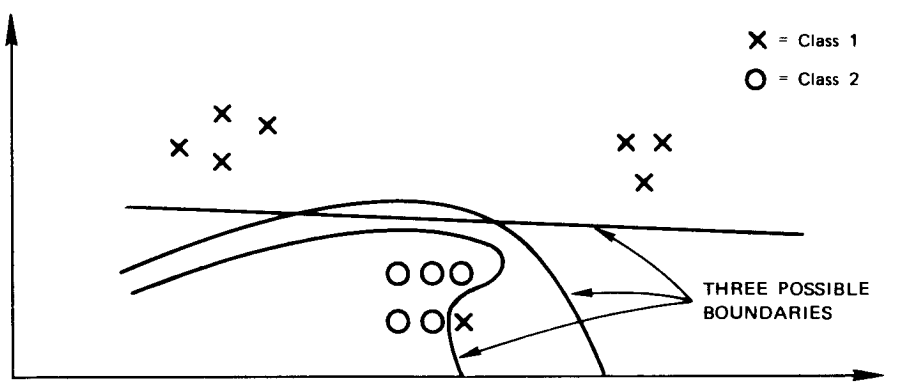
10. BOUNDARIES

Boundaries between classes can be specified implicitly by using the positions of the classes or explicitly by parametrizing the boundaries themselves. Boundaries can be lines, planes, or higher-order curves and surfaces. (Cooper, 1964.)

The amount of data available affects the generalizability of estimates of both boundary and class location parameters. (See Allais, 1966.) When the number of objects is small relative to the number of parameters to be estimated, estimated parameters of the boundaries are greatly affected by each object in the specific data set. As a result, the boundaries may not be generally applicable, since they tend to be primarily a function of the specific data set used in estimating the parameters that determine the class boundaries (Figure 11).



(a) MINIMUM SQUARED ERROR CLUSTERING PARTITION BOUNDARIES DEFINED IMPLICITY BY CLUSTER CENTERS



(b) EXPLICITY–DEFINED DISCRIMINATION BOUNDARIES WITH LINEAR, QUADRATIC AND HIGHER ORDER BOUNDARIES SHOWN

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FIGURE 11 BOUNDARIES

11. OPTIMALITY

There is almost always a criterion expression, usually mathematical, associated with classification that expresses the user's notion of what a classiciation should accomplish.

11.1 Criteria of Optimality

Some typical criteria that have been used in classification are listed in Table II.

Table II

TWO CRITERIA OF OPTIMALITY

Clustering	Sum of Squared Distance $C = \sum_{i=1}^{N} \sum_{j=1}^{v} \left(x_{ij} - {}_{i}K_{j}\right)^2$ where ${}_{i}\vec{K}$ is the cluster center associated with the i^{th} object and N is the number of objects.
Discrimination	Percentage of error $= 100 \left[1 - \left(\frac{\text{objects incorrectly classified}}{N}\right)\right]$

Perhaps the most common criterion in clustering is the sum of squared distances within a cluster or its complement, the sum of distances of the cluster centers from the overall average, where each cluster center is weighted by the number of objects in the cluster. In discrimination the criterion of errors of classification generally is

used; sometimes this criterion is generalized to indicate overlap between various pairs of classes.

Other criteria include functions of the within and between scatter matrices (see Friedman and Rubin, 1967 and Appendix A); entropy (Sebestyen, 1966); similarity within to similarity between (Bonner, 1964); coefficient of belongingness (Fortier and Solomon, 1966); and "total entropy" (Rogers and Tanimoto, 1960).

11.2 Procedures for Seeking Optimal Partitions

Having indicated that it is necessary to provide a clustering technique, either implicitly or explicitly, with some criterion for determining group membership, let us discuss the kinds of techniques by which people have tried to find optimal partitions. There are four general approaches to finding optimal partitions:

(1) Total enumeration of all possible partitions and an evaluation of a criterion for each possible partition. In sorting data into classes we need to consider the following factors: (a) the number of partitions into which the data are to be sorted, (b) the basis on which to do the sorting, and (c) the numbers of objects there are in a data base. In clustering we cannot consider, even for moderate-sized data sets, all possible partitions of the data set into clusters. For example, if we divide a group of 100 objects into 10 clusters of 10 objects each, we can find that there are approximately 10^{82} different partitions to be considered. Most techniques are designed to find good partitions of the data without examining all possible partitions.

(2) Limitations on the number of possible partitions, as for example in hierarchical clustering or stepwise procedures.

(3) Heuristic procedures that do not examine all possible partitions but rather select out partitions for evaluation that are presumed to be the more significant partitions.

(4) Theoretical procedures that test only undominated partitions; the key to this approach is the ability to specify which partitions dominate.*

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*For example, any partition that minimizes the sum of the squared distances of objects to their respective cluster centers must be contiguous. W. D. Fisher (1958) discusses this. In general, not much is known about this approach.

12. SEVEN CLASSES OF TECHNIQUES

We have indicated a number of characteristics that can be used to separate various discrimination and clustering techniques into classes. In fact, we can classify techniques so that for each type of discrimination technique we are able to find a similar clustering technique. (The reverse does not seem to be true, in that several clustering techniques seem to have no counterpart discrimination technique.)

We can now move toward specification of algorithms. We shall attempt to characterize techniques in terms of the category or cluster description used, and with a rough description of the sorting procedure. For some classes of techniques we shall indicate for which types of problems that class seems most appropriate. We also consider the amount of divergence a technique can handle, the amount of variety for which it is suitable, the amount of computation involved, and underlying assumptions about the structure of the data.

We divide the techniques in two sets in Table III. The first column of the table gives the name for a clustering technique; the second column gives the name for the discrimination procedures corresponding to that class of clustering techniques.

12.1 Factor Analysis and Discriminant Analysis

There is an extensive literature on factor analysis. For an excellent introductory article see Rummel (1967). Briefly we note that principal components--one method of obtaining new composite variables by rotation of the axes of the measurement space--seeks to find directions

Table III

CLASSIFICATION PROCEDURES

Clustering*	Discrimination
factor analysis (Rummel, 1967)	discriminant analysis (R. A. Fisher, 1936)
clumping (Sokal and Sneath, 1963)	nearest neighbor (Cover and Hart, 1967)
partitioning (Friedman and Rubin, 1967)	mode seeking (Rosen and Hall, 1966)
decomposition of mixtures (Spragins, 1966)	decision theory (Chernoff and Moses, 1959)
corridor methods (Bledsoe, 1963)	linear adaptive machines (Nilsson, 1965)

*Miscellaneous clustering techniques include: total enumeration (Fortier and Solomon, 1966); graph-theoretic (Zahn, 1969); potential functions (Meisel, 1968); function-oriented techniques (Gitman, 1970); accumulation (gravity-analog) approaches (Butler, 1969); clustering about lines (Eusebio and Ball, 1968); and interactive clustering (Hall, Ball, and Wolf, 1969).

of maximum variation in a set of data. Principal components analyzes variation in objects without regard to category.

Discriminant analysis seeks to find directions that maximize variation between categories while at the same time taking into account intra-class covariance structure. Discriminant analysis seeks to separate as much as possible the average vectors of the various different categories by producing composite dimensions in which these means have maximum variation. The discriminant directions are the principal components of the between-category covariance matrix as modified by the (pooled) within-category covariance matrix (cf. Figure 12) [cf. (Fisher, 1936), (Rao, 1948), (Anderson, 1951)].

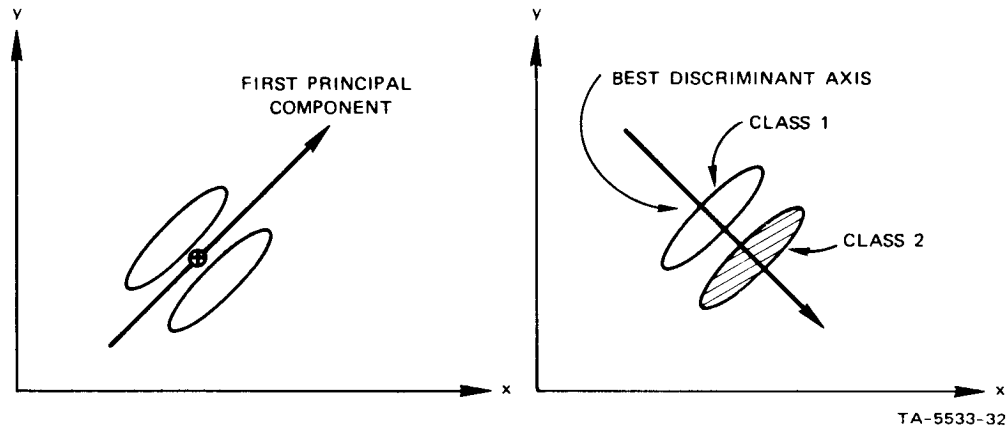

FIGURE 12 FACTOR ANALYSIS AND DISCRIMINANT ANALYSES

Both methods are most appropriate where the data set (in clustering) or individual categories (in discrimination) are adequately characterized by the mean and covariance matrix. Principal components is particularly useful in obtaining a smaller number of composite variables from a large number of original variables.

12.2 Clumping and Nearest Neighbor Techniques

In both clumping and nearest neighbor techniques a class is described by a list of its members. Objects are sorted on the basis of closeness to individual patterns. Conceptually, these are the simplest of the clustering and discrimination techniques. Clumping techniques have been used widely in numerical taxonomy and, as commonly used, assume hierarchical data organization (cf. Figure 13).

In clumping, distance can be measured to a cluster in terms of either the closest member in the cluster, the average member in the cluster, or the farthest member in the cluster. Clumping techniques become quite similar to partitioning techniques as the class description shifts from using the closest or farthest member in the set to using the

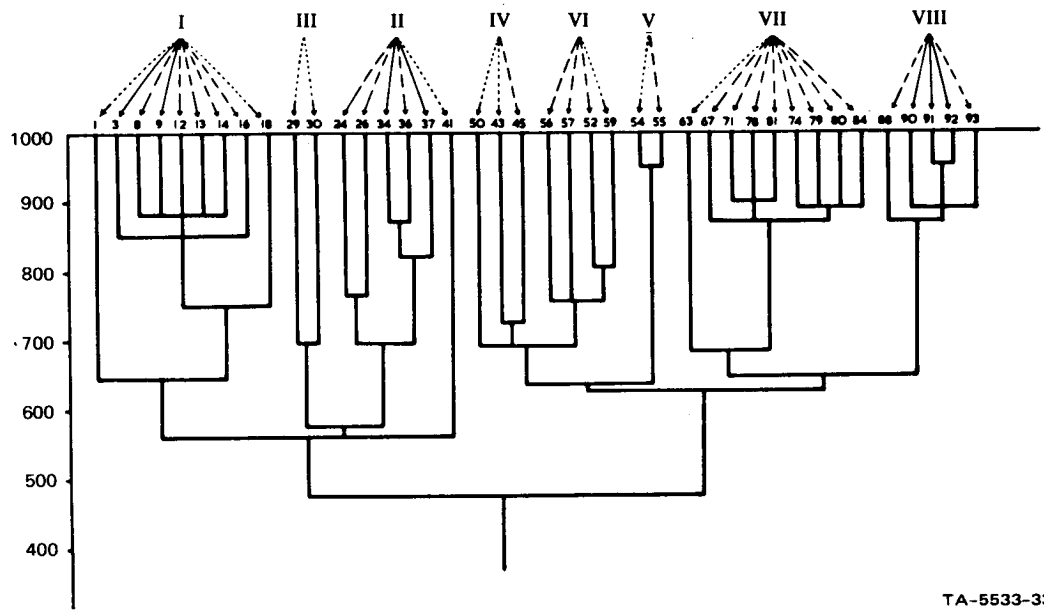

FIGURE 13 HIERARCHICAL DENDROGRAM PLOT DISPLAYING THE RESULTS OF A CLUMPING PROCESS

average of the objects in the cluster. However, clumping procedures do not allow nonhierarchical groupings. Sokal and Sneath (1963) discuss clumping in the context of numerical taxonomy in biology. Ward (1963), Johnson (1967), and Hartigan (1967) give additional insights.

Clumping procedures generally consist of a procedure of the following form:

(1) Gather the data and establish a data matrix.

(2) Calculate a similarity or distance matrix between all pairs of objects.

(3) Join the best or closest pair of patterns into a cluster.

(4) Join the next closest pair of objects. This can include the joining of a single object to the cluster created in a previous merger.

(5) Continue this process until all objects are together in a single cluster. To accomplish this requires a procedure for joining two clusters of patterns.

In clumping, objects are never reshuffled--the hierarchical form is always maintained. For example, in procedures that agglomerate objects, once two objects have been placed in the same cluster they must remain in that cluster together. This drastically reduces the number of alternative ways in which objects can be combined. (The binary decomposition of the entire set requires consideration of partitions equal in number to the sum of binomial coefficients of K things taken two at a time, rather than products of such coefficients.)

The procedure for nearest neighbor techniques consists of the following (Figure 14):

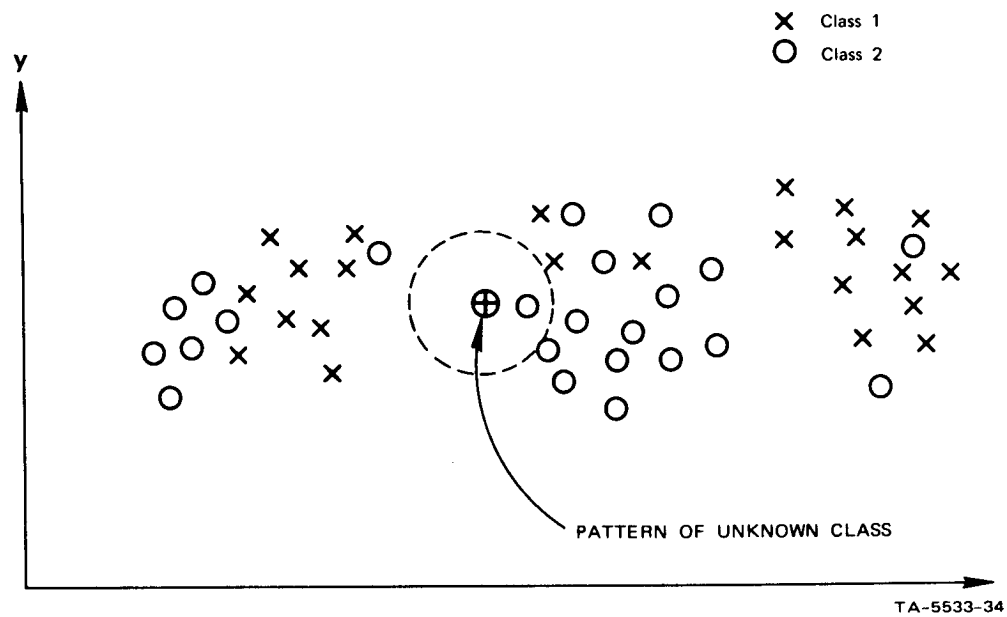

FIGURE 14 NEAREST NEIGHBOR TECHNIQUE

(1) Determine the closest K objects to the object that is to be classified.

(2) The unknown object is assigned to that category having a plurality of members in the set of K nearest neighbors. The nearest neighbor class description uses only category assignments for each object.

Detailed analysis of the power of nearest neighbor techniques is possible (Cover and Hart, 1967). Modifications of the simple procedure include removing "interior objects" from the class description, since interior objects do not affect discrimination between various categories.* The nearest neighbor procedure in essence provides an empirical probability distribution, with the probability mass being concentrated at each of the objects.

12.3 Partitioning and Mode Seeking

Partitioning and mode seeking techniques describe a cluster or category in terms of an average position for each cluster or mode (Figure 15).

One category can have several modes. Variation around that average can also be retained in the class description. The point of view for these techniques comes primarily from statistics, as opposed to probability theory. Partitioning can be divided into:

(a) Sequential Threshold Partitioning--Here a cluster center is selected and all of the data objects within some threshold distance of that center are placed in that cluster.

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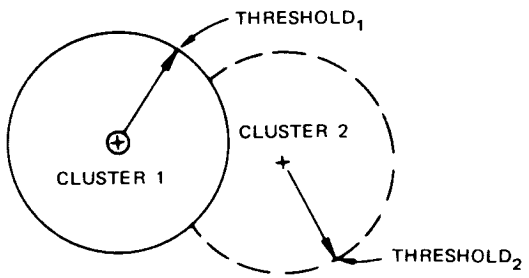
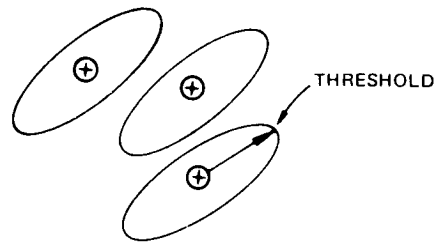
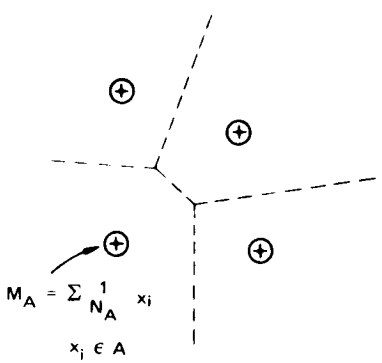
*Nearest neighbor techniques have been tried with rather mixed results, which suggests that they may be more sensitive to the particular data structures than some other discrimination techniques. Note that interior objects do affect discrimination when k-nearest neighbors are used.



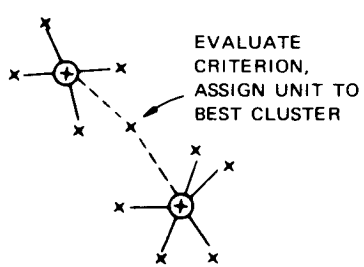
(a) SEQUENTIAL THRESHOLD PARTITIONING



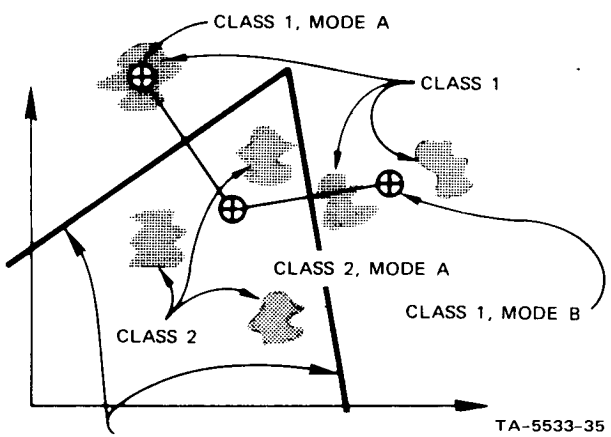
(b) PARALLEL THRESHOLD PARTITIONING



(c) PARALLEL PARTITIONING



(d) OPTIMIZING PARTITIONING



(e) MODE-SEEKING BOUNDARY BETWEEN CLASSES

FIGURE 15 PARTITIONING AND MODE-SEEKING TECHNIQUES

(Sometimes the threshold used depends on the compactness of the data.) These objects are removed from the data set and a second cluster center is chosen, etc. (Rogers and Tanimoto, 1960.) Figure 15(a).

(b) Parallel Threshold Partitioning--In this method a number of cluster centers are chosen and a threshold value is set to determine the partitioning. Objects are introduced one at a time and associated with a cluster center. The location of that cluster center with which the data object is associated is then modified, usually by updating a running average, as each object is added and the process is repeated, usually with a change in the threshold. (Sebestyen, 1966.) Figure 15(b).

(c) Parallel Partitioning--As in parallel threshold partitioning, several cluster centers are chosen. The entire data set is then partitioned into disjoint subsets. Using the subsets new cluster centers are determined, etc. (Forgy, 1963; Ball and Hall, 1967a.) Figure 15(c).

(d) Optimizing Partitioning--The sequence of steps in a typical optimizing partitioning approach is described below. (Friedman and Rubin, 1967.) Figure 15(d).

The ISODATA algorithm for parallel partitioning was discussed in Section 1.1. An algorithm for optimizing partitioning might consist of the following steps:

(1) Collect data and set up a data matrix, perhaps using principal components to produce dimensional parsimony.

(2) Either consider all data as one cluster and then select any object as a nucleus for a second cluster or arbitrarily partition the data into several groups.

(3) Take each object, one at a time--and usually in a fixed sequence--and tentatively transfer that object to each of the other clusters while evaluating the change in a criterion.

(4) Transfer an object to the cluster for which the criterion most improves.

(5) Continue transferring objects until the criterion can no longer be improved. A stopping point is guaranteed to exist.

(6) Increase or decrease the number of clusters.

(7) Repeat the procedure for a new number of clusters.

It may be useful to select new starting partitions on the basis of either a random partitioning of data or a reassignment of some selected objects to a new cluster, which can include forcing objects into a cluster.

Mode seeking operates in a manner similar to partitioning, but it uses labelling information to influence the creation of new modes. Simply stated, a new mode for a category is created only when objects within that category are closer to a mode of a different category than they are to modes in their own category. Mode seeking techniques are most useful when a class has several modes, with objects from other classes lying in between the modes. The Rosen and Hall technique discussed above in Section 1.2 is an example of such a technique.

12.4 Decomposition of Mixtures and Decision Theory

In both decomposition of mixtures and decision theory, the category or cluster is described by a probability distribution. In the case of decomposition [Figure 16(a)], the family of a distribution is assumed and distributions of this form are fitted to data [cf. (Alens, 1967), (Spragins, 1966), (Stanat, 1968), (Patrick and Hancock, 1967), (Sammon, 1968a), and (Blischke, 1965)].

In decision theory the distributional form also is assumed, but the labelling allows a user to estimate the parameters of the distribution directly* [Figure 16(b)]. [See, for example (Abramson and Braverman, 1962) in which the category description is updated sequentially.]

Decomposition techniques seem most appropriate where the degree of overlap among underlying clusters is assumed to be high, and where a paradigm provides sufficient guidance to support assumptions regarding the presumed form of the probability distribution of the data. Divisions found by decomposition are often similar to those found by partitioning when overlap between groups is small. Decomposition techniques are by far the most elegant mathematically, and yet often it is not clear whether

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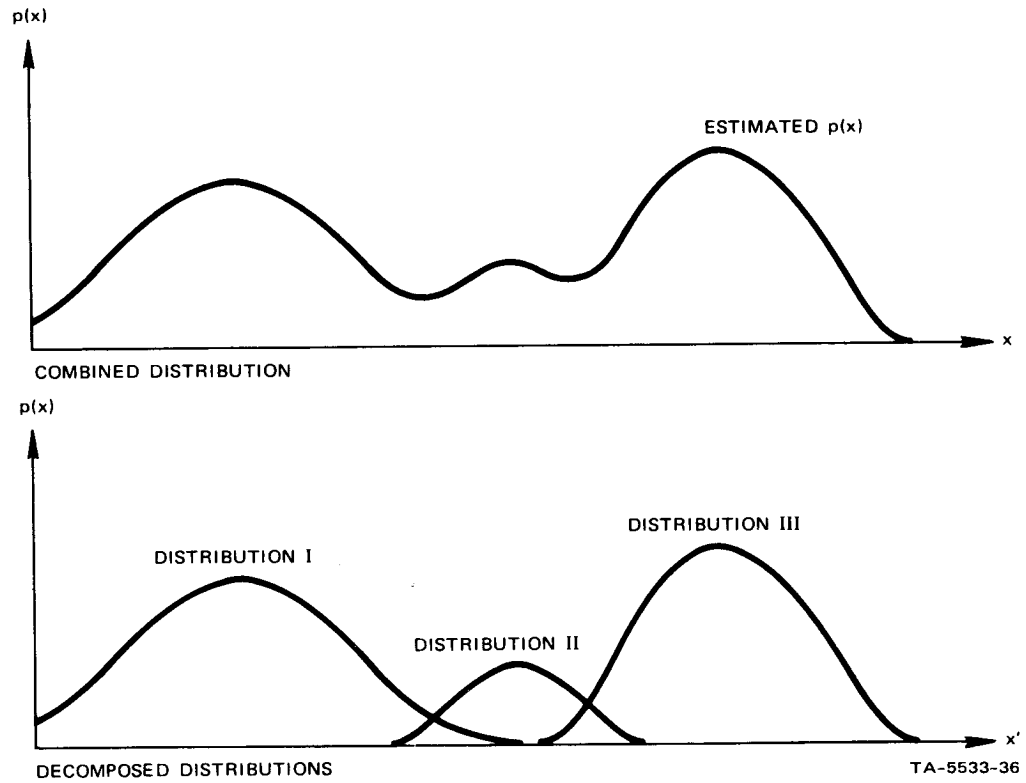
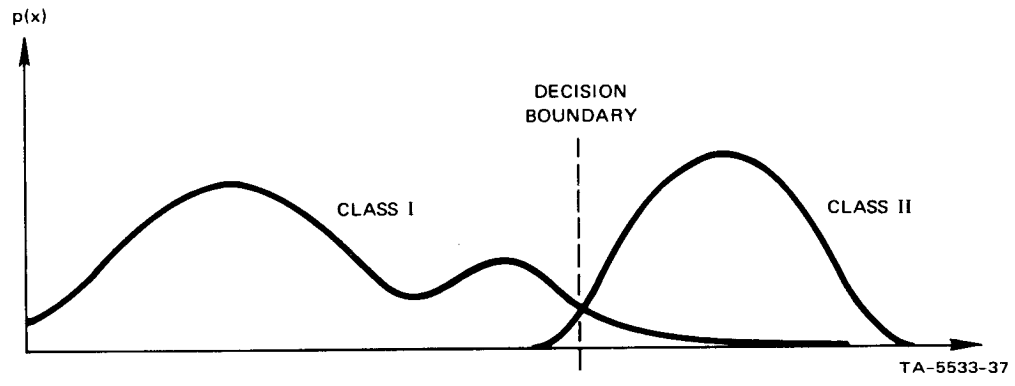
*One interesting aspect of decision theory concerns the construction of empirical distribution functions. Specht (1967) suggests that a Parzen window (Parzen, 1962) can be centered at each data sample, which, in effect, spreads the probability at each object more broadly and thereby permits a smoother characterization of the distribution of a given category of objects. On small data sets that smoothing unfortunately performs badly in the tails beyond the extremal sample objects, where the smoothed distribution takes on the characteristics of the window function. Sebestyen made an interesting combination of mode-seeking and decision theory by breaking his data set up into modes within each category and then characterizing each mode by a normal distribution with a diagonal covariance matrix.



(a) DECOMPOSITION OF MIXTURES



(b) DECISION THEORY

FIGURE 16 DECOMPOSITION AND DECISION THEORY

the number of objects available to a user provides estimates of the distributional parameters sufficiently accurate to support the rigor of the techniques.

12.5 Corridor Methods and Linear Adaptive Machines

In both of these techniques a class is described by characterizing the boundaries between the clusters or categories (Figure 17).

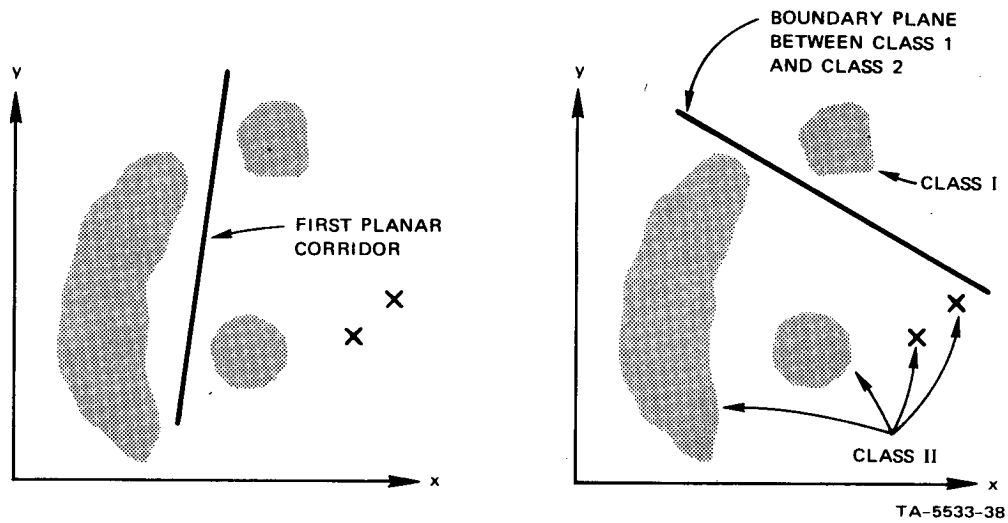


FIGURE 17 CORRIDOR METHODS AND LINEAR ADAPTIVE MACHINES

Corridor methods (Bledsoe, 1963) attempt to pass a plane through the data such that data are maximally distant from the plane while still having the plane cut somewhere through the middle of the data.

In the case of linear adaptive machines (Nilsson, 1965), the goal is to pass the plane between classes so that all objects in one class are separated from all objects in other classes. This approach attracted

much attention in pattern recognition, where it is referred to as a learning machine.

12.6 Miscellaneous Techniques

There are several techniques that do not fall neatly into any of the previous categories. In some cases they represent significant departures from the procedures mentioned above.

12.6.1 Total Enumeration

Conceptually, one could evaluate all possible partitions of a set of objects into clusters. This approach is considered in Fortier and Solomon (1966). However, the direct calculation is not practical for data sets exceeding 20-30 objects. Fortier and Solomon tried sampling partitions but discovered that the distribution of the criterion they used is skewed so that one has a much higher probability of finding a poor partition than one might expect. On identical data it was found that even a simple clumping procedure, such as that of King (1967), produces partitions as good as those produced by any reasonable amount of computation using total enumeration.

12.6.2 Graph Theoretic Approaches

Innovations include some clustering approaches based on graph theory [(Bonner, 1964), (Abraham, 1964)]. Independently, sociologists developed techniques for organizing matrices obtained from sociometric data (Spilerman, 1966).

Two recent papers (Zahn, 1969) and (Gower and Ross, 1969) point out that the structure of data can be brought out by connecting objects that are close in a minimal way, that is, by using what in graph

theory is called a minimal spanning tree. They point out that one can examine a minimal spanning tree and detect good links at which to cut a tree into various branches without having to assume a particular distributional form for the structure of the data.*

12.6.3 Function-Oriented Techniques

Gitman (1970) describes a technique in which the density of objects in a region around each object is used to indicate the relative importance of that object. Each object receives a value for its "characteristic function" that depends on the number of objects lying within a distance T of that object. Objects then are ordered by the value of their characteristic function, and clusters are built one object at a time in an ordered sequence around those objects having the highest values [cf. also (Rogers and Tanimoto, 1960), (Sawrey, Keller, and Conger, 1960)]. While it is not apparent from the examples given by Gitman how his technique handles various shapes of data, it is clear that his approach can handle skewed distributions, since assignment to clusters is not primarily a symmetrical function of distance, but rather focuses on local object density and ordering of distances.

12.6.4 Clusterings around Lines and Line Segments

Most clustering techniques use the notion of a point as the center of a cluster. Two techniques exist that fit line segments or curves to a set of objects (Figure 18) [Sneath, 1966) and (Eusebio

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*It was not clear, however, from Zahn's report (Zahn, 1969) how expensive computationally these procedures are for even moderately large data sets with more than two significant principal components.

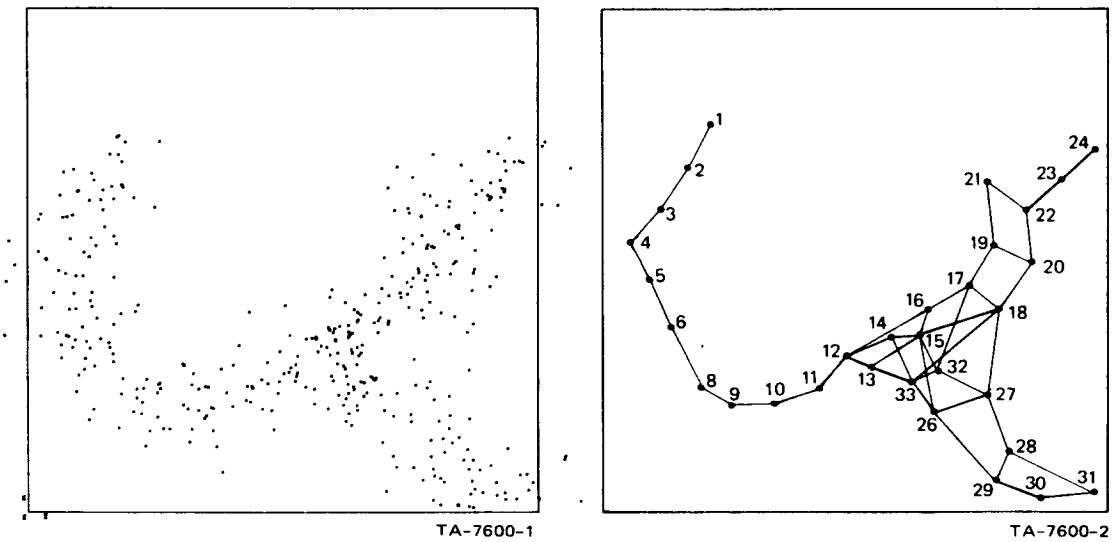

FIGURE 18 CLUSTERING ABOUT LINE SEGMENTS

and Ball, 1968)]. These go beyond standard regression techniques in that data can be fitted with branched curves.*

12.6.5 Gravity Procedures

Clustering can occur if a set of objects is allowed to collapse, as if under the force of "gravitational" attraction between objects, where the force is a function of distance from the other objects (Figure 19) [(Ihm, 1965) and (Butler, 1969)].

At each step the force on each object is computed and direction of movement is determined. Each object is allowed to move a user-determined incremental distance along this direction and then a

*It is not clear how the minimal spanning tree technique (Zahn, 1969) compares to these two approaches. It appears that the minimal spanning tree approach will be more sensitive to individual objects since the other approaches position the lines by averaging together the effect of several objects.

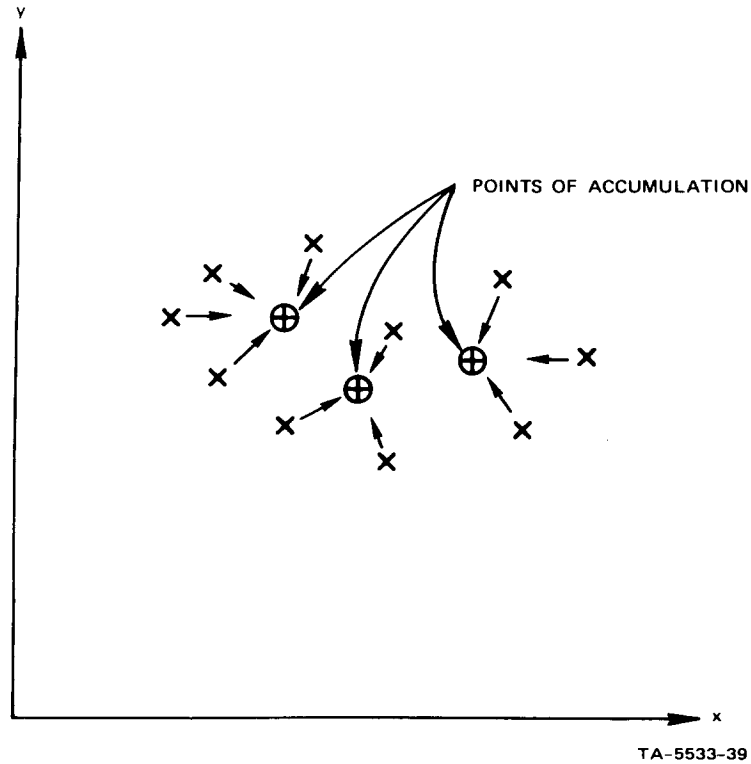

FIGURE 19 GRAVITY TECHNIQUES

new force of attraction is determined. Eventually, object points tend to coalesce at centers of accumulation, which are considered to be cluster centers.

12.6.6 Interactive Clustering

A reader by now might feel that he could learn the structure of his data more effectively by interactively controlling the clustering and discrimination algorithms. This is the goal of the PROMENADE computer system. The PROMENADE system (Ball and Hall, 1967a) was developed around the ISODATA clustering program and the Rosen-Hall mode-seeking discrimination program. It uses an interactive computer system

with a graphic display in order to provide the researcher with a variety of graphic plots and considerable data manipulation capability, as well as control over clustering and discrimination routines.

At present, available commands for interactive clustering allow modification of ISODATA process parameters and lumping or splitting clusters specified by the user at the end of each iteration. Commands planned include freezing or unfreezing clusters (and the objects associated with that cluster), freezing or unfreezing individual objects, and selecting specific variables to be used in a particular iteration of the clustering without constructing a new data set.* It would be useful to be able to change the weighting given particular variables and to change from hard boundaries, such as one gets from partitioning techniques, to soft boundaries, such as those found in decomposition techniques (cf. Wolfe, 1967).†

12.7 Multidimensional Scaling Techniques

Users of classification techniques should be aware of multidimensional scaling procedures. These scaling techniques can interrelate objects and variables, provide an estimate of the "number" of dimensions

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*We have found that the waveform and link-node plots provide most of the information a user needs in order to control clustering. For example, link-node plots allow us to examine intracluster and intercluster distance structures. It is important to provide a user with zoom capabilities that allow him to look at a macroscopic overview immediately before and after inspecting a microscopic distance structure. Waveform plots allow him case in seeing which particular variables most directly determine clustering. With a pointer a user could then pick variables he wishes to use in a clustering procedure in a particular iteration.

†See Section 14 for a discussion of hard versus soft boundaries.

needed, produce a two- or three-dimensional space in which to view the relationships between cluster centers, or provide an alternative to principal components in producing a new set of variables of lower dimension. As importantly, multidimensional scaling provides the reader with a method of using ordinal indications of similarity to determine metric distances between objects.

Shepard (1962) provides one of the earliest approaches, with Kruskal (1964a, b) providing an important modification to Shepard's approach. Guttman (1968) summarizes a long series of investigations on "smallest space analysis." Torgerson (1968) looks at implications of multidimensional scaling for examining similarity structures, while Green et al. (1968) discuss applications to market analysis.

13. INTERPRETATION AND EVALUATION OF THE OUTPUT FROM CLASSIFICATORY TECHNIQUES

Clustering and discrimination can be viewed as part of a sequential decision process in which each run of a clustering or discrimination program provides additional information. Although users of computer programs are accustomed to mathematical and tabular output, additional forms and modes of expressing results can greatly facilitate the interpretation of output from classification programs. Our experience with PROMENADE leads us to believe strongly that powerful data manipulation capabilities should be well integrated into the researcher's computer system. The system should allow a user convenience in using the output of one routine as part of the input into another routine. [Tukey and Wilk (1966) have a number of wise words on this subject.]

13.1 Clustering and Discrimination Computer Systems

Several clustering and discrimination computer-based systems* have been developed to date. Interactive graphic computer systems, such as

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*Details regarding the PROMENADE system can be obtained from David Hall of Stanford Research Institute, Menlo Park, California. In addition to the PROMENADE system mentioned in Section 13.1, there is the BCTRY system, developed by Professors Tryon and Bailey, which combines factor analysis and clustering techniques with plotting procedures. The BCTRY system is available from Professor Bailey, Department of Psychology at the University of Colorado, and it is described in Tryon and Bailey (1966). Other such systems include: OLPARS, developed at Rome Air Development Center (Sammon, 1968b); a system at Purdue University (Patrick, 1969); and a system developed at General Motors by Stanley, Lendaris, and Nienow (1967). These systems are discussed in Ball and Hall (1970). Sources of other classification programs are listed in Appendix B.

PROMENADE, open interesting avenues toward team efforts, whereby data analysts and subject specialists work on the data alongside each other in order to exchange the specialized knowledge each has. The data analyst conveys how a technique operates. The subject specialist indicates what is theoretically or empirically significant in the data (cf. Ball and Hall, 1970).

A graphic computer facility for classification could operate either in a time-shared environment or in a multiprogramming environment.

Such a facility should eventually include most of the following:

(A) Software

(1) Extensive data-manipulation facilities (perhaps similar to those in the WILBER system at Stanford University and in ADMINS at MIT) that would provide:

- The capability to sort and select data for detailed analysis--potentially from a large data base.
- The ability to tag data and to add information such as the size of the residuals to a vector of data values.
- Convenience from the user's viewpoint to encourage multiple looks and use of the information from one type of technique to understand the results obtained from using another technique. This includes the ability to visually inspect the raw data.

(2) A wide variety of mathematical/graphical techniques including at least the following analysis techniques:

- Regression
- Multiple table creation and manipulation capabilities
- Clustering techniques for grouping objects
- Factor analysis, including principal components
- Discriminant analysis, including new techniques developed within "pattern recognition"
- Analysis of variance
- Other, more standard techniques in the BIOMED package (Dixon, 1967)
- Many plotting procedures carefully linked to the analysis programs.

All of the techniques included should allow varying degrees of user control over the technique.

(3) Line-by-line compilation and special languages for analysis program writing, perhaps interactively, by analysts. The programs so written should be insertable in the system and accessible to that user within the system (i.e., he can pass his own data to and from his own program). This provides ways for the individual user to create programs that meet his special needs while still using the power of the larger system.

(B) Hardware

(1) Mass graphic capability, such as microfilm recorders, that allows the user to indicate conveniently what he wishes done. For example:

- All possible scatter plots (or selected scatter plots) on the set of data that he is considering.
- The construction of movies from multiple plots.

(2) Refreshed CRT displays for dynamic viewing of the data.

(3) Slave storage tubes for multiple views of his data. These tubes are increasingly available and less expensive. Some contain the capability of providing a xerox copy of what is seen on the display.

(4) Use of the mouse* rather than the light pen for pointing.

(C) System Configuration

(1) Multiprogramming or time-sharing capability with a large machine so that a relatively powerful analysis process can be used.

(2) Use of a small machine to do some local analyses, to regenerate a display, and serve as message processor for the large machine.

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*A simple device with two right-angle potentiometers that allows the user of a computer system to indicate locations on a CRT to a computer.

13.2 Graphic Modes of Presentation

Graphic presentation of data can aid in interpreting results. Link-node plots, waveform plots, scatter plots, and residual plots have all been helpful in evaluating classification. Accessibility and ease of use of the plots are important considerations.

The link-node plot provides a convenient way of displaying the distances between the cluster centers (Figure 20). The link-node plot relates directly to the graph theoretic approach using minimal spanning trees (Zahn, 1969).

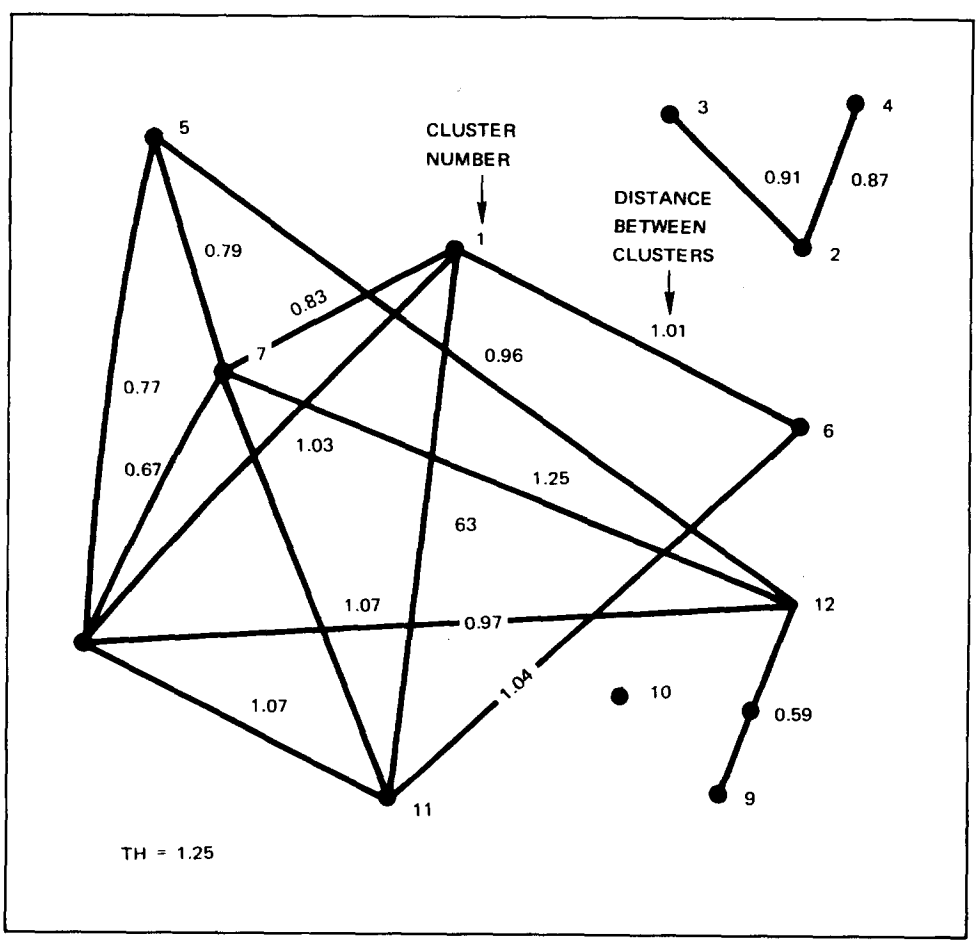

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FIGURE 20 LINK-NODE PLOT

A waveform plot allows a user to see relationships across all of his variables (Figure 21). Over many objects it can indicate subsets of variables and of objects that cluster together. For example, some objects may cluster on one subset of variables, while other objects cluster on a different subset. Plots showing the distribution of values over variables for each of the clusters or categories are also useful diagnostic aids.

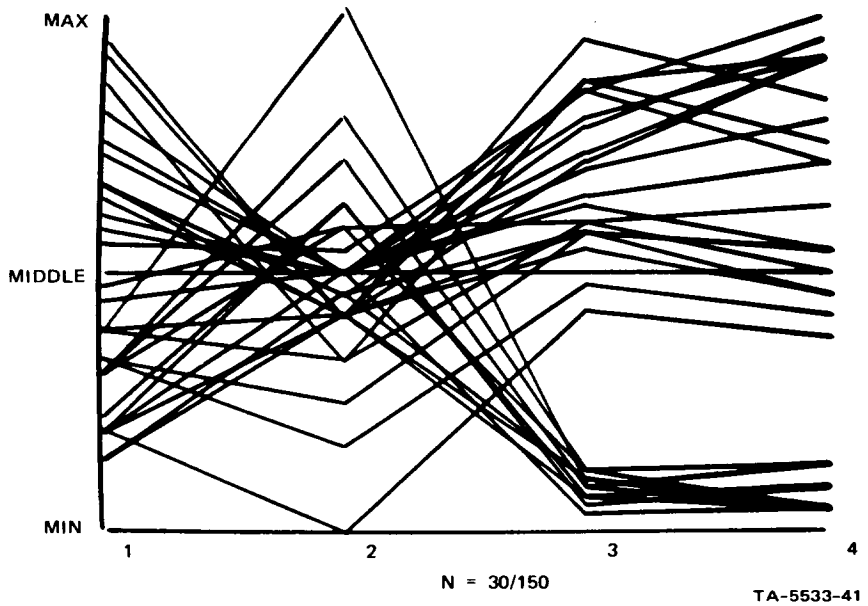

FIGURE 21 PROFILE OR WAVEFORM PLOT

Confusion tables show the number of correct discriminations and indicate the number of objects incorrectly placed in a category for each category. They can indicate to a user those subsets in his data causing his program most difficulty in arriving at a good discrimination sorting rule. (Clustering within category is also useful here.)

The rearrangement of matrices, such as the similarity matrix, the distance matrix, or a matrix giving numbers of variables differing

significantly between pairs of clusters (see Table IV), can bring similar rows and columns together and so make the structure of the matrix more obvious.* The rearrangement of the matrix in Table IV is shown in Table V.

Table IV

NUMBER OF VARIABLES DIFFERING SIGNIFICANTLY

BETWEEN COMBINATIONS OF CLUSTERS

Clusters	1	2	3	4	5	6	7
1	X	1	0	2	2	4	4
2	1	X	1	5	1	2	5
3	0	1	X	2	1	2	4
4	2	5	2	X	4	5	1
5	2	1	1	4	X	1	5
6	4	2	2	5	1	X	5
7	4	5	4	1	5	5	X

5 Variables:

1. Percent unemployed
2. Percent unskilled
3. Percent poor
4. Percent nonwhite
5. Percent with two or more cars.

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*In larger matrices a simple program that systematically rearranges rows and corresponding columns in order to optimize $|i - j|A^2(i,j)$, where i and j give the row and column and A(i,j) the value, will accomplish such a rearrangement.

Table V

TABLE IV REARRANGED TO SHOW HOW

REARRANGEMENT INCREASES CLARITY

Clusters	6	2	5	3	1	4	7
6	X	2	1	2	4	5	5
2	2	X	1	1	1	5	5
5	1	1	X	1	2	4	5
3	2	1	1	X	0	2	4
1	4	1	2	0	X	2	4
4	5	5	4	2	2	X	1
7	5	5	5	4	4	1	X

5 Variables:

1. Percent unemployed
2. Percent unskilled
3. Percent poor
4. Percent nonwhite
5. Percent with two or more cars.

Squared-error curves plot the sum of squared distances of objects from their respective cluster centers against the number of cluster centers (Figure 22). A sharp drop in this curve usually indicates that there exists strong clusteredness in the data. Singleton (1968) was able to show the curve to be star-shaped, which is a weak form of convexity. (A star-shaped curve is convex with respect to its end points and any single interior point.) Sum-of-squared-error curves have been calculated for uniform random data and for simplex data as a function of the number of dimensions and the number of clusters (Hall, Tepping,

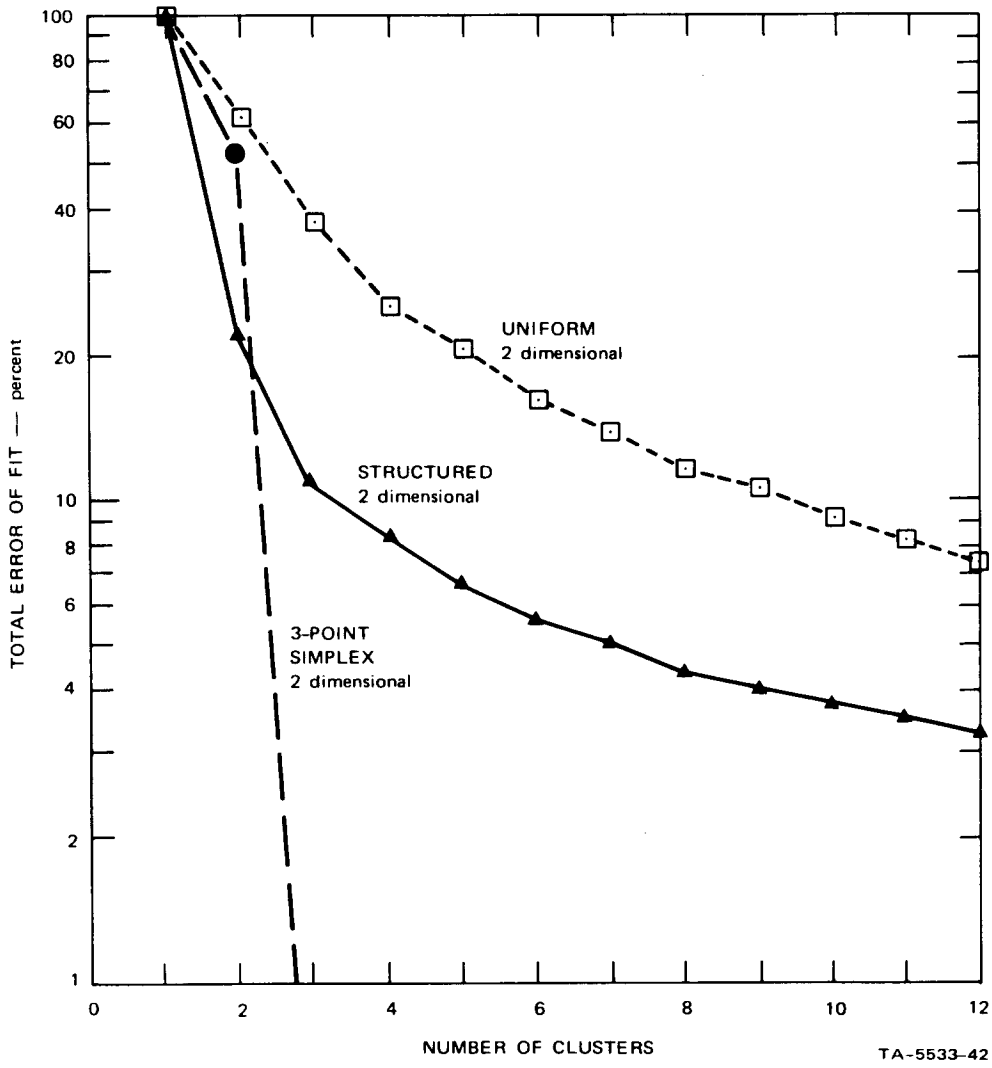


FIGURE 22 SUM OF SQUARED ERROR CURVE

and Ball, 1970). These curves allow an investigator to judge how much his data depart from uniform random data and simplex data.

Decomposition of summary curves and other descriptive statistics is a powerful procedure in interpreting the output of classification techniques. For example, given a sum-of-squared-error curve, we can

decompose the total sum into the contribution of each variable to the total. This decomposition is plotted in the case of some economic data about Indian cities in Figure 23. Unlike the smoothly decreasing curve that represents a sum of squared errors for all of the variables combined, a curve for an individual variable seldom decreases smoothly but instead exhibits sudden large decreases, usually in concert with several other variables. This grouping of variables is useful in indicated interactions between subsets of variables.

13.3 Summary Statistics

Criteria of optimality mentioned in Section 11.1 can be used as descriptive statistics of the clustering. Within-cluster compactness as it relates to the between-cluster difference is indicated by some of these criteria. Distance tables giving the pairwise distances between all modal centers can also be quite useful, as can the size of residuals in indicating how well we are fitting our data. The level of significance tables for conventional tests of significance do not apply in the case of clustering, since the assumptions of such tests are not satisfied. These tests, however, can be used to give useful information about cluster separation or about the amount any given variable has affected the clustering.

13.4 Tests for the Structure of the Data

Several tests for the structure of data have been suggested. MacQueen (1967) compares values of various descriptive statistics for original data with the same statistics calculated on data randomized in the following way. Values of the objects for each variable are rearranged randomly within each variable, providing data having the same

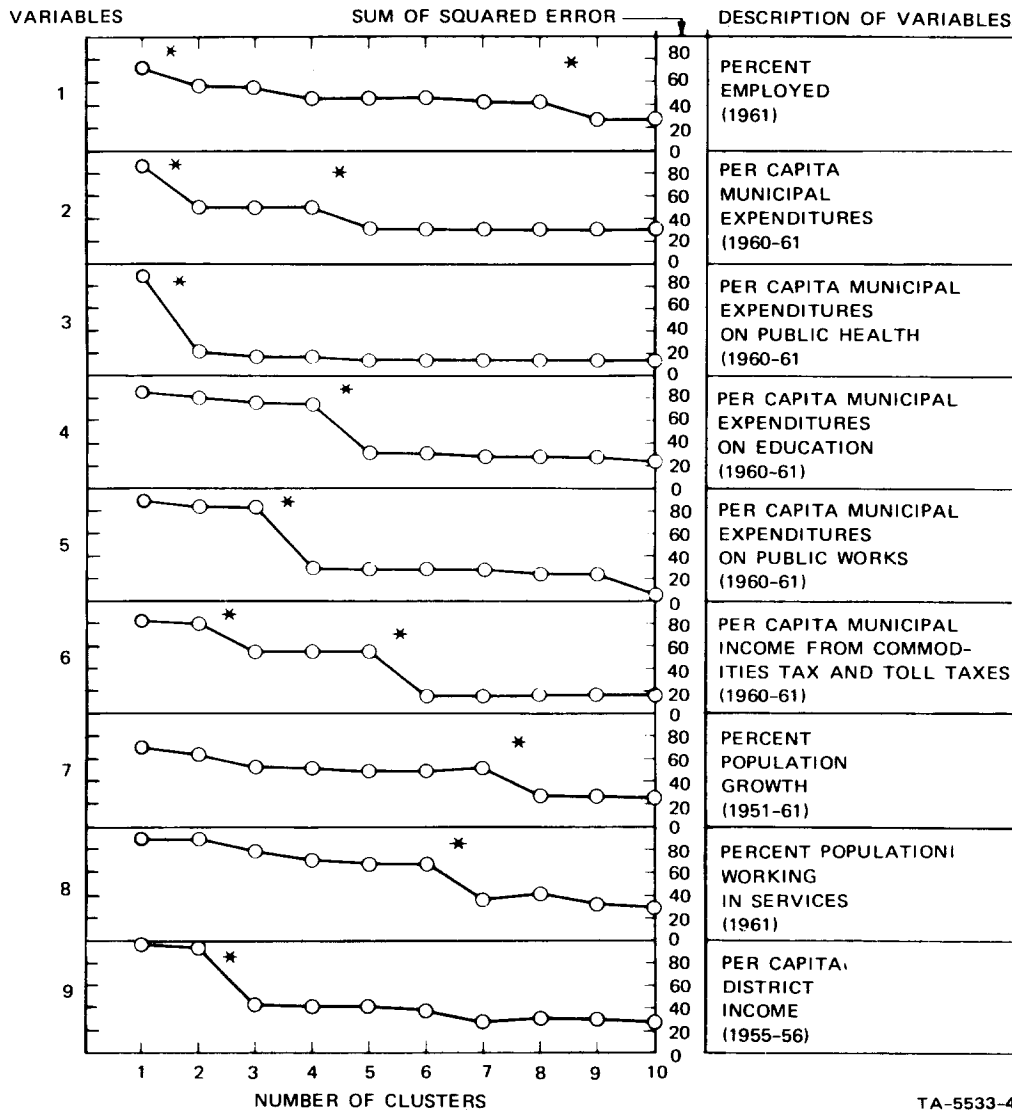


FIGURE 23 DECOMPOSITION OF SUM OF SQUARED ERROR

marginal (individual variable) distributions but having different multivariate relationships between variables. If data are randomized several times, it may be possible to develop regions of confidence for clustering into each number of clusters and thus infer that the original data do or do not have "significant" clustering with respect to randomized data (Figure 24).

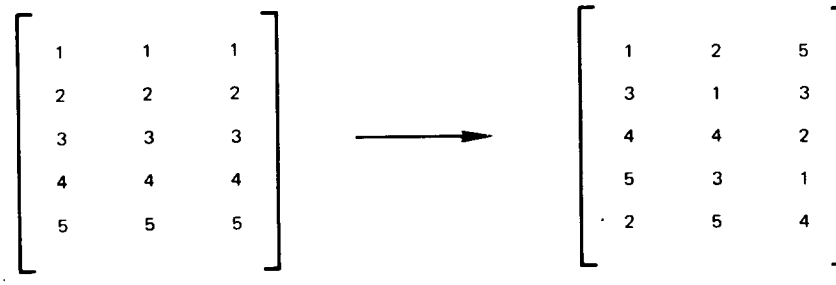
Another alternative is to look for sensitivity of clusters to the presence or absence of both variables and objects. If we find that the deletion of one variable dramatically affects the clustering, we have a good indication that the variable deleted is a variable that is important in determining the clustering.

13.5 Useful Test Data

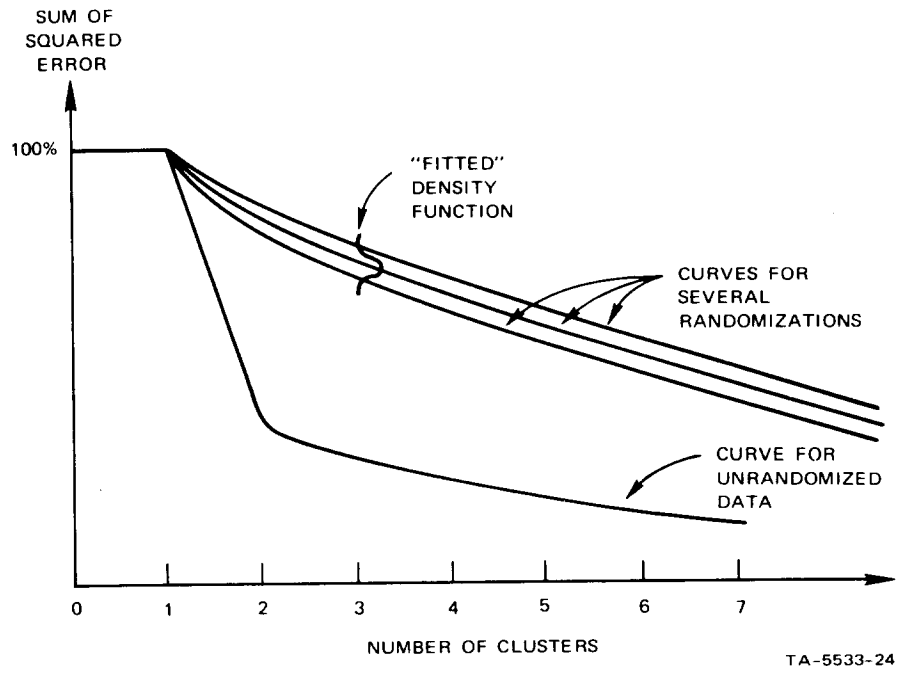
In addition to statistical tests of data, it is useful to compare algorithms on a variety of data sets. Data sets of known structure are presented in Figure 25.

Data in Figure 25(a), which has two large-spread subsets and two small-spread subsets, are useful in detecting sensitivity to cluster variability as opposed to data contiguity. The second data set [Figure 25(b)], in which a compact set of data is embedded within a set of much lower density, can test sensitivity to variations in data density such as might be extracted by a decomposition technique.

Anderson-Fisher iris data, which is essentially a two-dimensional data set, can be separated quite nicely into its three categories even without the explicit use of category information, i.e., by clustering



(a) REARRANGEMENT OF VALUES WITHIN COLUMNS



(b) PLOT OF SUM OF SQUARED ERROR CURVE FOR RANDOMIZED AND UNRANDOMIZED DATA

FIGURE 24 "TESTS FOR SIGNIFICANCE" OF CLUSTERING

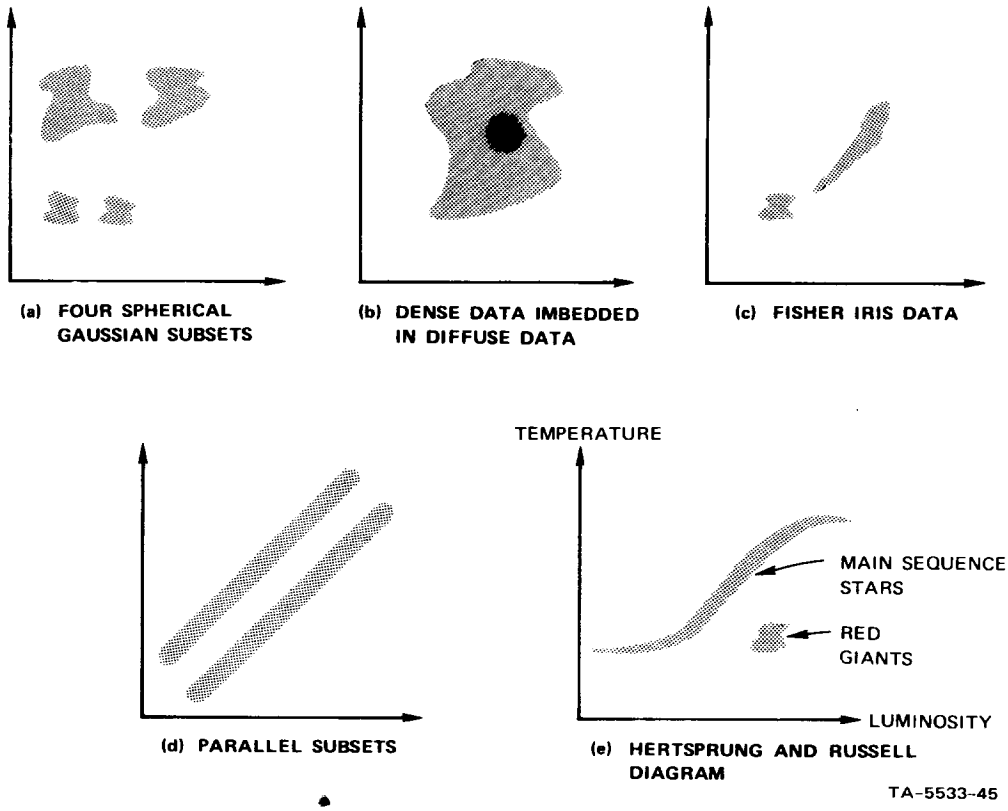


FIGURE 25 USEFUL TEST DATA

[Figure 25(c)]. It is included here because of its wide usage as a test set.*

The fourth data set [Figure 25(d)] tests whether a technique is able to detect separation between data sets that is small relative to the length of each of the clusters.

---

*The value of the iris data may well be limited to exposition of a technique. Because the set is two-dimensional, it does allow cluster centers or discrimination boundaries to be plotted. Effective sorting relative to the taxonomic categories implies that the technique used finds something close to a partition that optimizes the determinant of the within- and between-cluster covariance matrix, as in Friedman and Rubin (1967).

The luminosity and temperature star data [Figure 25(e)] provide a scientifically significant data set in which two markedly different cluster shapes are found.

13.6 Other Comments

In some data sets the number of variables is nearly the same as the number of objects. Using clustering or discrimination procedures to sort objects from such data sets can be misleading, since the degrees of freedom also increase as the number of variables increases and the classification rules become too sensitive to the specific objects being sorted. For example, in discrimination one may find that errors in testing a discrimination rule increase as the number of measurements increases (Allais, 1966). One procedure that has been used successfully to reduce the number of variables is principal components analysis, which usually uses correlations between variables as a basis for linearly combining variables into composite variables (cf. Section 12.1). Other alternatives include increasing the number of objects by further observation or arbitrarily excluding some variables. Examination of the sum of the residuals from a partial fit of a set of cluster centers to a set of data can indicate the number of composite measurements needed to describe variations in the data as discussed in the text related to Figure 23. Principal components retains as much of the squared variation as possible for a given number of composite variables.*

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*See Friedman and Rubin (1968) for an excellent and extensive analysis of a set of psychiatric data using clustering, discrimination, and principal components.

14. CLASSIFICATION AND COMPUTER MODELS

Classification provides a suggestive vantage point from which to view some social science phenomena, for it suggests some alternatives to models that have been used to date. With the advent of interactive computing machinery, it seems possible to entertain the use of rather complex models. In what follows we discuss some speculations and indicate a form that an interactive model might take.

Consider, for example, Figure 26(a), in which we show objects centered around one mode. The average variation, even when the data are felt to be of one class, may not be particularly instructive. Using two or more centers [Figure 26(b)] allows the variation of the various dimensions in each cluster to differ. By increasing the number of classes a user can sometimes escape making bland statements about variation in relationship to an overall class average and can instead make more penetrating comments about each of several subclasses and the relevance of the variables associated with each subclass.

Consider the rejection or acceptance of information by a person. In Figure 26(c) we conceive of the rejection of information by an individual as depending on his degree of uptightness around his "conceptual cluster centers." We characterize the degree of uptightness by a threshold around concepts held by that individual. As the person grows more uptight, much of the information he is receiving falls outside the

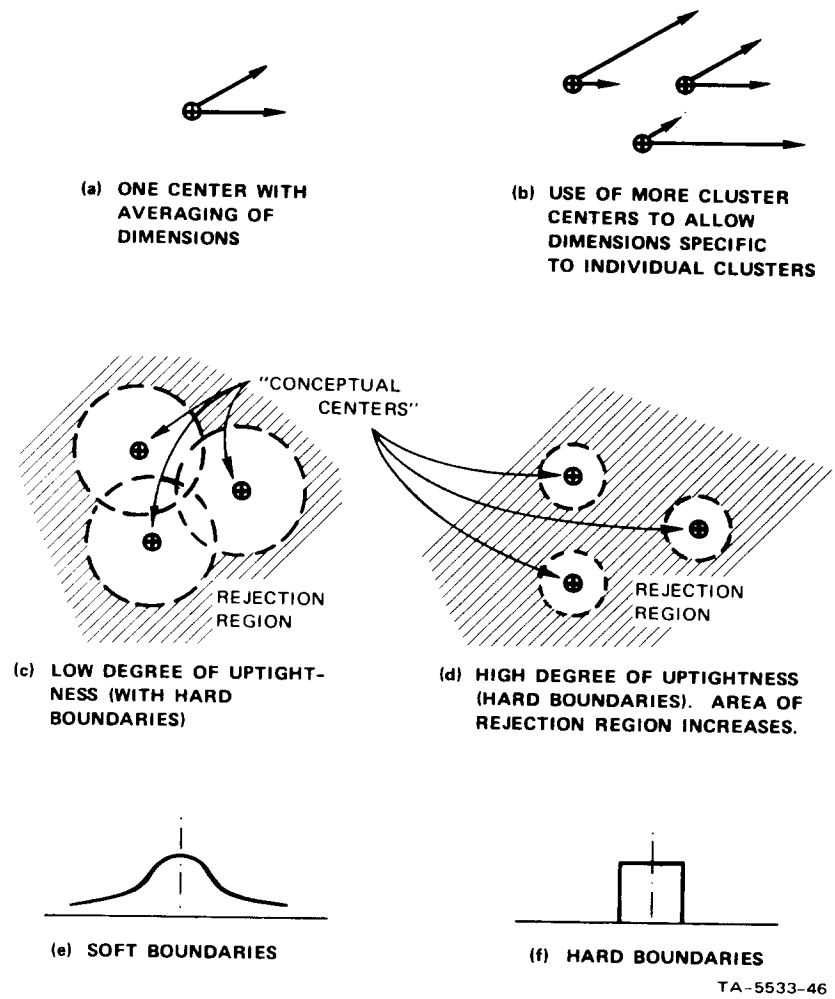

FIGURE 26 CONCEPTS FROM CLASSIFICATION TECHNIQUES AS MODEL COMPONENTS FOR SOCIAL SCIENCE MODELS

regions he associates with each concept--into a region of rejection or regions perceived as dissimilar, as suggested in Figure 26(d).*

In yet another description, a person is seen as having soft boundaries or hard boundaries around his "conceptual centers" [Figures 26(e) and 26(f)]. If a person has hard boundaries, something either is or is not similar to a given concept. A person having soft boundaries sees a piece of information as varying in the degree of relevance to various concepts that are already internalized.

The concepts and descriptive structure provided by clustering that we have mentioned above, combined with our experience on interactive graphic computers, causes us to think that it would be possible to provide social scientists with an interactive computer-based system for storing and examining dynamic processes. This system would complement classification systems like PROMENADE and would provide a means of capturing details obtained through detailed data analysis. Figure 27 portrays features of such a system having three levels with which the user interacts.

The first level provides a process view, in which a user views a dynamic representation of the process in which he is interested with the rate at which the process proceeds under his control. As he learns to interpret the symbols, he will be able to detect when the process thus simulated is not proceeding as he might expect, which would then cause him to examine the second level of the model.

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*Boredom, for example, would result from having tight boundaries around what is considered relevant information. When a person is bored, much of the information he perceives will appear to be irrelevant; it will, then, fall into a rejection region. In a different social context such information might produce fear rather than boredom.

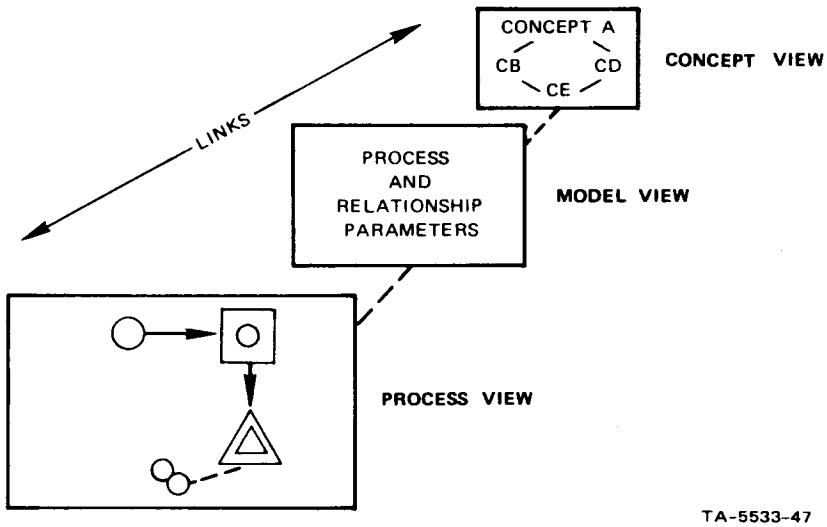


FIGURE 27 DYNAMIC PROCESS GRAPHIC COMPUTER MODELING SYSTEM

Programs controlling the process view are accessible at the second level. At this level the user has access to the model that governs the process seen at the first level. The system would do the bookkeeping necessary to link the user to that portion of his model that is most influencing the simulation process at any given time. In this way, the researcher can correct portions of the computer-held model that depart from his expectations. A user would be able to modify processes and the parameters of the model rapidly, whereupon he could view the process again at the first level to see whether the process depicted more nearly matches his intuition.

There is a third level in the system, which might be called the concept view, in which concepts, experimental evidence, and other related information are accessible. A user would be able to pull out information relevant for the particular part of the process that he is interested in modifying.

The above three facets of the system, once linked together, constitute an ambitious prototype of future interactive modeling systems. One major aim would be to allow the user to move from one level to another easily, retrieving and modifying pertinent information, while allowing the observer operator to use his intuition. Criticism could be directed to details in the model, rather than to some general overview. The consequences of changes in the model could be quickly seen. It is clear that there are many complexities in attempting to bring such a model into existence. It probably would press the state of the art, both in social scientific disciplines and in computer technology; however, we believe it is definitely in the realm of the possible--and even in the realm of the not too distantly possible.

15. CONCLUDING REMARKS

There exist a number of similarities and parallels between clustering and discrimination. The kind of data and the particular research problems determine which technique is appropriate. Clustering and discrimination should be viewed not as isolated techniques, but in relation to the entire spectrum of data analysis techniques. The solution to a complex sorting task frequently requires a complex sorting rule. Simple solutions may prove more misleading than helpful.

From our present perspective, it appears that there is a need for more effective computer languages that allow for construction of clustering or discrimination procedures appropriate to a specific research task. The potential of clustering and discrimination procedures in combination with other techniques, such as regression, needs further exploration. Moreover, succinct algorithmic languages are needed in order to provide a common language for communicating the nature of various classification algorithms among researchers in many fields.

The reader should keep in mind that an explicit choice of one technique does not lead to objective Truth. Explicitness does make it easier to see the assumptions and the weightings that a researcher employs in reaching his conclusions.

Appendix A

SUMS OF SQUARES AND T, W, AND B

We discuss in various places the sums of squares as a criterion by which to judge clusters or as a means to weight distance. In what follows we describe in more detail how sums of squares are usefully defined and calculated, and how the notion can be extended to "scatter matrices" having similar properties.

Consider a set of objects divided into three classes: X, Y, and Z. There is to be only one variable observed. If the object belongs to X, the object will be denoted as x; if in Y, y; if in Z, z. Suppose there are N(x) objects in X; N(y) in Y, N(z) in Z. The means of the variables in the groups are:

$$\bar{x} = \frac{1}{N(x)} \sum_{i=1}^{N(x)} \left(x_i\right)$$

$$\bar{y} = \frac{1}{N(y)} \sum_{i=1}^{N(y)} \left(y_i\right)$$

$$\bar{z} = \frac{1}{N(z)} \sum_{i=1}^{N(z)} \left(z_i\right)$$

where i labels the i^{th} object in that class.

The grand mean of the values for the single variable over all classes is:

$$C = \frac{N(x)\bar{x} + N(y)\bar{y} + N(z)\bar{z}}{N(x) + N(y) + N(z)} \quad .$$

The total sum of squares of all values about the grand mean is, by definition:

$$T = \sum_{i=1}^{N(x)} \left(x(i) - c\right)^2 + \sum_{i=1}^{N(y)} \left(y_i - c\right)^2 + \sum_{i=1}^{N(z)} \left(z_i - c\right)^2 \quad .$$

Within each class the sum of squares about the group means is, by definition:

$$W(x) = \sum_{i=1}^{N(x)} \left(x_i - \bar{x}\right)^2$$

$$W(y) = \sum_{i=1}^{N(y)} \left(y_i - \bar{y}\right)^2$$

$$W(z) = \sum_{i=1}^{N(z)} \left(z_i - \bar{z}\right)^2 \quad .$$

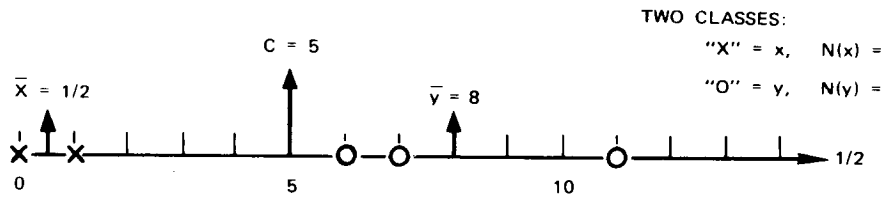
The sum of squares between groups is, by definition:

$$B = N(x)(\bar{x} - c)^2 + N(y)(\bar{y} - c)^2 + N(z)(\bar{z} - c)^2 \quad .$$

It follows that $T = W + B$. (This is true for any partitioning of the data.) An example of this calculation is given in Figure A-1.

Among alternative partitionings of the objects, it is reasonable to rank partitions in ascending order of the value of T/W, i.e., the total variance divided by the within-group variance.

To generalize to more than one variable, suppose there are V variables and N objects and let $X'(i,j)$ denote the value of the j^{th} variable on the i^{th} object. Let P(i) (a normalized vector)



$$T = \sum_{i=1}^{2} (x_i - c)^2 + \sum_{i=1}^{3} (y_i - c)^2$$

$$= [(0 - 5)^2 + (1 - 5)^2] + [(6 - 5)^2 + (7 - 5)^2 + (11 - 5)^2] = 82.$$

$$W(x) = \sum_{i=1}^{2} (x_i - \bar{x})^2 = (0 - 1/2)^2 + (1 - 1/2)^2 = 1/2.$$

$$W(y) = \sum_{i=1}^{3} (y_i - \bar{y})^2 = (6 - 8)^2 + (7 - 8)^2 + (11 - 8)^2 = 14.$$

$$W = W(x) + W(y) = 14\ 1/2.$$

$$B = N(x)\ (\bar{x} - c)^2 + N(y)\ (\bar{y} - c)^2$$

$$= 2 \times (1/2 - 5)^2 + 3 \times (8 - 5)^2 = 40\ 1/2 + 27 = 67\ 1/2.$$

$$T = W + B \text{ or } 82 = 14\ 1/2 + 67\ 1/2.$$

This equation extends nicely to more dimensions courtesy of the Pythagorean Theorem since dimensions are independent with respect to addition of sums of squares.

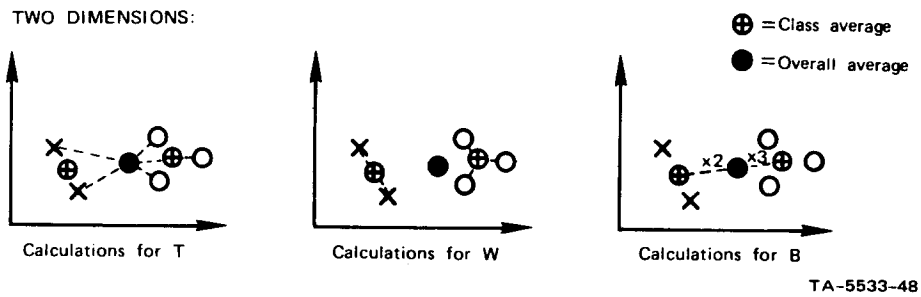


FIGURE A-1 AN EXAMPLE OF THE CALCULATION OF SUMS OF SQUARES

$$P(i) = [X(i,1), \ldots, X(i,V)]$$

denote the row vector of observations on the V variables of the i^{th} object. Suppose each $X'(i,j)$ is normalized in terms of a deviation of the i^{th} object's j^{th} variable about the grand mean $\bar{X}'(\cdot,j)$ of the j^{th} variable divided by the standard deviation of that variable over all of the objects in all classes; i.e., if $X(i,j)$ is the normalized row observation, then

$$X(i,j) = \frac{X'(i,j) - \bar{X}'(j)}{\sqrt{\frac{1}{N}\sum_{i=1}^{N} [X'(i,j) - \bar{X}'(i)]^2}} \quad .$$

Let there be G groups with N(k) objects in each: K = 1, ..., G so that:

$$\sum_{k=1}^{G} N(k) = N \quad .$$

Let P[k(i)] be the (row) vector of variables of the $k(i)^{th}$ object of the k^{th} group. Let C(k) be the (row) vector of variable means within the k^{th} group; i.e.,

$$C(k) = \left[\frac{1}{N(k)}\sum_{i=1}^{N(k)} P[k(i),1], \ldots, \frac{1}{N(k)}\sum_{i=1}^{N(k)} P[k(i),V]\right] \quad .$$

Then corresponding to the within-group sum of squares for one variable (V = 1), there is the V × V within group covariance matrix for the K^{th} group:

$$W(k) = \sum_{i=1}^{N(k)} \Big(P[k(i)] - C(k)\Big)' \Big(P[k(i)] - C(k)\Big)$$

and for the total within-group sum of squares

$$W = \sum_{k=1}^{G} W(k) \quad .$$

The T is the V × V scatter matrix (normalized around the grand mean)

$$T = \sum_{i=1}^{N} P_i' P_i \quad .$$

An example of the calculation of a scatter matrix, the corresponding covariance matrix, and the correlation matrix is shown in Figures A-2 and A-3. Note that if the data consist of two widely separated clusters and if the data are partitioned into two clusters in the obvious way, then most of the variation in T is due to B.

The Euclidean distance between the i^{th} and k^{th} objects with respect to all V variables is

$$D^2(i,k) = \left(P_i - P_k\right)\left(P_i - P_k\right)' = \sum_{j=1}^{V} \left(P_{ij} - P_{kj}\right)^2 \quad .$$

Note that a V-element row vector times a V-element column vector is a scalar, that is, an ordinary number.

The so-called Mahalanobis distance is

$$D^2_{mah}(i,j) = \left(P_i - P_j\right)W^{-1}\left(P_i - P_j\right)' \quad .$$

This is also a scalar. Note that the choice of the groups may affect W and hence change D^2_{mah}.

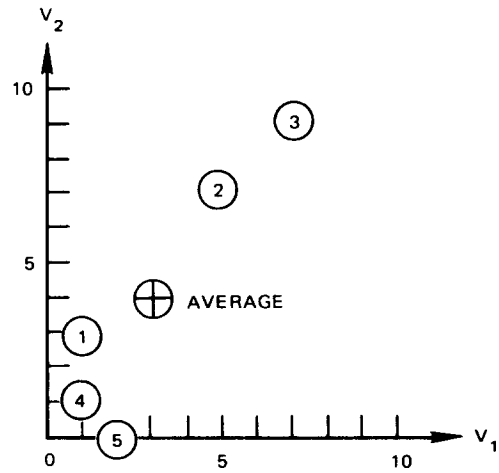


FIGURE A-2a THE SCATTER PLOT OF THE DATA

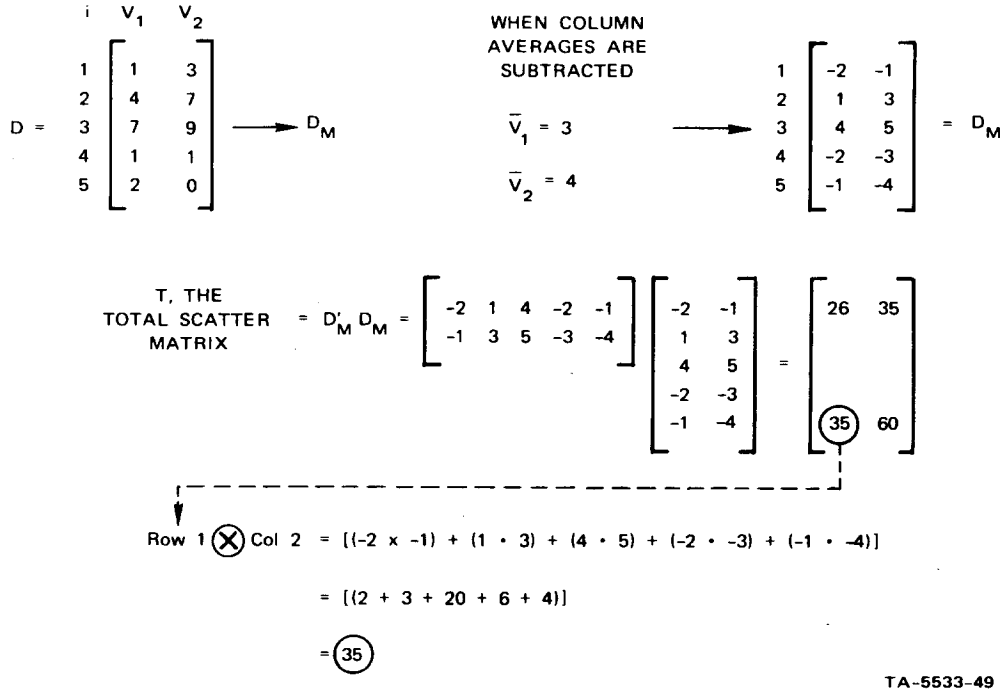


FIGURE A-2b THE CALCULATION OF THE SCATTER MATRIX

$$K, \text{ COVARIANCE MATRIX} = \left[T/(N-1)\right] = \begin{bmatrix} 26/4 & 35/4 \\ 35/4 & 60/4 \end{bmatrix} = \begin{bmatrix} 6.50 & 8.75 \\ 8.75 & 15.00 \end{bmatrix}$$

FIGURE A-3a CALCULATION OF COVARIANCE MATRIX FROM T OF FIGURE A-2b

NORMALIZE TERMS BY $\left(\frac{k_{ij}}{\sqrt{k_{ii} \cdot k_{jj}}} \right)$

THEN GET THE CORRELATION MATRIX

$$R = \begin{bmatrix} 6.5/6.5 & 8.75/\sqrt{6.5 \times 15} \\ 8.75/\sqrt{6.5 \times 15} & 15/15 \end{bmatrix}$$

$$= \begin{bmatrix} 1.00 & 0.89 \\ 0.89 & 1.00 \end{bmatrix}$$

TA-5533-50

FIGURE A-3b CALCULATION OF CORRELATION MATRIX FROM COVARIANCE MATRIX OF A-3a

To get good groupings, one might maximize the ratio of the determinants of T and W,

$$|T|/|W| \quad .$$

This index is invariant under nonsingular linear transformations of the variables.

We can write

$$W^{-1}T = I + W^{-1}B \quad .$$

If λ_i are the roots of the determinantal equation

$$|B - \lambda W| = 0$$

where λ is the V × V diagonal matrix whose diagonal elements are λ_i, then maximizing

$$|T|/|W|$$

is equivalent to maximizing

$$\prod_{i=1}^{V} (1 + \lambda_i) \quad .$$

This is related to the problem of maximizing

$$\text{Trace } W^{-1}B = \sum_{i=1}^{V} \lambda_i \quad .$$

Appendix B

SOME AVAILABLE CLUSTERING AND DISCRIMINATION PROGRAMS

Systems

BCTRY. See Tryon and Bailey (1966) for details.

Contact:
Prof. Daniel Bailey
Department of Psychology
University of Colorado
Boulder, Colorado

(Programmed for a CDC 6400 computer among others.)

PROMENADE (Ball and Hall, 1970)

Interactive clustering with independent control over the subsets in the data and easy selection of subsets of variables and patterns.

Contact:
Mr. David J. Hall
Stanford Research Institute
Menlo Park, California 94025

OLPARS

For discrimination and clustering with a focus toward designing machine systems for pattern recognition.

Contact:
Dr. John W. Sammon, Jr.
NCS Computing Corp.
Computer Symbolic Div.
310 E. Chestnut Street
Rome, New York 13440

Programs

A variety of clustering programs exist. If you wish to obtain a program, here are some of the places where they can be obtained:

CLUMPING TECHNIQUES

NUMERICAL TAXONOMY System

A complete numerical taxonomy system, including a number of useful related programs

Contact:
Prof. James Rohlf
State University of New York
Stoney Brook, Long Island, New York

HYCLUS (See Johnson, 1967 for details.)

Contact:
Steven C. Johnson
Bell Telephone Laboratories
Murray Hill, New Jersey

PARTITIONING TECHNIQUES

Optimizing Partitioning for Minimum Squared Error

Contact:
Dr. Richard Singleton
Stanford Research Institute
Menlo Park, California 94025

Friedman and Rubin Algorithm (Friedman and Rubin, 1967)

Available through the IBM Share system for an IBM 360-67

DECOMPOSITION TECHNIQUES

NORMIX and NORMAP. See Wolfe (1969) for details.

Contact:
Dr. John Wolfe
U.S. Naval Personnel Research Laboratory
San Diego, California

MISCELLANEOUS

Density Cluster Analysis

A program for compressing a multivariate histogram

Contact:
G. T. Nygreen
Office of Survey Research and Statistical Studies
O-S-11, Green Hall
Princeton University
Princeton, New Jersey

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HOTELLING'S WEIGHING DESIGNS

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I. FORMULATION OF THE WEIGHING PROBLEM

0. Introduction

Hotelling's weighing problem [29] took its origin in a casual illustration furnished by Yates [45]. The weighing problem has sinced attained a distinctive growth with its roots and branches in different directions. Over the years, the foundation of the problem has been paved by the contributions made by Kishen [31], Mood [33], Kempthorne [30], Rao [39], Plackett and Burman [35], Banerjee [1-16], Raghavarao [36-38], Zacks [46], Beckman [18], Sihota and Banerjee [40], Dey [21], Kulshreshtha and Dey [32], and possibly others.

This is a review article on the weighing problem, and in this article the problem has been traced from its origin and discussed briefly with an indication of the main developments.

As far as known to the author, all papers which have a direct bearing on the problem have been cited in the list of references. If any paper has not been mentioned, it is because the author is not aware of the contribution.

1. Origin of the Problem

The Original Example of Yates

In furnishing an illustration of "independent factors" in *complex experiments*, that is of factors which do not interact, Yates [45] considered the following problem: A chemist has seven light objects to weigh, and the scale would require a zero correction. The obvious technique would have been to weigh each of the seven objects separately, and to make an eighth weighing with no object on the scale so that the zero correction could be determined. Thus, to determine the weight of each object, one would take the difference between the readings of the scale when carrying that object and when empty. Assuming that *systematic errors are non-existent* and that *the errors are random*, the standard error of each weighing may be denoted by σ, and the variance by σ^2. On these assumptions, the variance of the estimated weight is $2\sigma^2$, and its standard error is $\sigma\sqrt{2}$.

As an improvement over the customary technique, Yates [45] suggested that the objects be weighed in (eight) combinations according to the following scheme:

Weighing No.	Objects Weighed
1.	$a+b+c+d+e+f+g = y_1$
2.	$a+b\ \ +d\ \ \ \ \ \ \ = y_2$
3.	$a\ \ +c\ \ +e\ \ \ \ \ = y_3$
4.	$a\ \ \ \ \ \ \ +f+g = y_4$
5.	$\ \ b+c\ \ \ \ +f\ \ \ = y_5$
6.	$\ \ b\ \ \ \ +e\ \ +g = y_6$
7.	$\ \ \ \ c+d\ \ \ \ +g = y_7$
8.	$\ \ \ \ \ \ d+e+f\ \ \ = y_8$

(1)

In the above scheme, each object is weighed four times in the different combinations. In the four weighings of a given object, every other object is included twice. The remaining four weighings, that is, weighings without the object, also include every other object twice. Calling the readings from the scale $y_1, y_2, \ldots, y_8$, the weight of any object a can be determined as

$$a = \frac{y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7 - y_8}{4} .$$

A like expression is obtained for the other objects. As will be evident from the above expression, the weight of any particular object is found by adding together the four equations containing it, subtracting the other four, and dividing the algebraic sum by 4. It will further be noticed that the bias cancels out in the algebraic sum.

As the variance of a sum of independent observations is the sum of the variances, the variance of a by this improved technique is $\sigma^2/2$, which is 1/4 that of the customary method.

2. Improvement Suggested by Hotelling

With reference to the above example of Yates, Hotelling [29] suggested that even a further improvement would be possible, if Yates' procedure was modified by placing in the other pan of the scale those objects not included in the weighing. Calling the readings $Z_1, Z_2, \ldots, Z_8$, we may write the scheme of weighing operations as given below (interchanging c and d).

$$
\begin{aligned}
a + b + c + d + e + f + g &= Z_1 \\
a + b + c - d - e - f - g &= Z_2 \\
a - b - c + d + e - f - g &= Z_3 \\
a - b - c - d - e + f + g &= Z_4 \\
-a + b - c + d - e + f - g &= Z_5 \\
-a + b - c - d + e - f + g &= Z_6 \\
-a - b + c + d - e - f + g &= Z_7 \\
-a - b + c - d + e + f - g &= Z_8
\end{aligned}
\qquad (2)
$$

From these equations,

$$
a = \frac{Z_1 + Z_2 + Z_3 + Z_4 - Z_5 - Z_6 - Z_7 - Z_8}{8} .
$$

A similar expression is obtained for each of the other unknowns. The variance of each unknown by this method is $\sigma^2/8$. The standard error is half that by Yates' method, or a quarter of its value by the direct method of weighing each object separately. Here also, the bias gets cancelled out.

It may be pointed out here that in the example furnished by Yates, only one pan of the scale was used for placing the objects. One pan is used when the balance is of the spring balance type. But, in the improvement suggested by Hotelling, both pans of the scale were used. This, however, is possible only in a chemical balance.

3. Statistical Model Characterizing the Problem

In the light of the above illustrations, Hotelling [29] gave a precise formulation of the problem of Weighing Designs.

The formulation as given by him may be transcribed as follows:

Results of N weighing operations to determine the weights of p light objects fit in to the general linear hypothesis model, $Y = X\beta + \varepsilon$, where Y is an $N\times 1$ random observed vector of the recorded weights; $X = (x_{ij})$, $i = 1,2,\ldots,N$; $j = 1,2,\ldots,p$, is an $N\times p$ matrix of known quantities, with $x_{ij} = +1$, -1 or 0, if, in the i^{th} weighing operation the j^{th} object is placed respectively in the left pan, right pan or in none; β is a $p\times 1$ vector ($p \leq N$) representing the weights of the objects; ε is an $N\times 1$ unobserved random vector such that $E(\varepsilon) = 0$ and $E(\varepsilon\varepsilon') = \sigma^2 I_N$.

Consistent with the signs that the elements x_{ij} can take, the record of the i^{th} weighing is taken as positive or negative, according as the balancing weight is placed in the right pan or left.

X will be called the "design matrix." When X is of full rank, that is, when [X'X] is non-singular, the least squares estimates of the weights are given by $\hat{\beta} = [X'X]^{-1} X'Y$, where X' is the transpose of X. The covariance matrix of the estimated weights is given by $COV(\hat{\beta}) = \sigma^2[X'X]^{-1} = \sigma^2 C$. c_{ii}, which is the i^{th} diagonal element of C, represents the variance factor for the i^{th} object.

4. Original Example of Yates and Hotelling Cast in the Frame of the Above Model

The original example of Yates [45] has eight equations involving seven variables, a, b, c, d, e, f, and g. In fact, there is one more variable. The eighth variable is due to the bias, taken as an additional object whose weight may be determined, and

this variable is common to all the eight equations. Similarly, in the example suggested by Hotelling, the eighth variable, the variable due to the bias, which is common to all the eight equations, is not explicitly mentioned. Thus, in both the examples, we have eight equations involving eight variables. The solutions indicated earlier are the solutions obtained by solving the two systems of linear equations, where the number of variables is the same as the number of equations.

When X is a square matrix of full rank, the least squares estimates are given by $\hat{\beta} = [X'X]^{-1}X'Y = X^{-1}Y$. On the other hand, if we solve for β in $Y = X\beta$, β would be obtained also as $X^{-1}Y$. This means that the least squares estimates are the same linear functions of the observed Y's, as the true parameters would have been of the true y's. If X is an orthogonal matrix in the sense that $[X'X] = N\, I_N$, then $\hat{\beta} = X^{-1}Y = [X'Y]/N$, the same as the solutions as given in Hotelling's example [29].

5. Two Types of Problems

Two distinct types of problems would arise in practice. One type would refer to the "spring balance" and the other to the "chemical balance". In the spring balance problem, the elements, x_{ij}, are restricted to assume values +1 or 0, while in the chemical balance, these elements will either be +1, -1 or 0.

6. Efficiency of a Weighing Design

A Weighing Design has been called, in general, the best, if

(i) each variance factor c_{ii} is the least, or

(ii) $\sum_{i=1}^{p} c_{ii}/p$ is the least, or

(iii) det.$|X'X|$ is maximum (equivalently, det. $|C|$ is minimum).

Sometimes, det.$|X'X|$ has, in this note, been referred to as det.$|A|$.

Wald [41] suggested criterion (iii) for experimental designs in general, while Mood [33] suggested criterion (iii) for Weighing Designs. Mood [33] pointed out further that other definitions of "best designs" might also conceivably be preferred. Problems might arise in which one might prefer to

(iv) minimize the variance factors subject to restriction that they be equal,

(v) minimize some function of the variance factors, or

(vi) minimize only a certain subset of the c_{ii} on a minor of the matrix (c_{ij}),

as might be the case when one wanted only rough estimates of the weights of some of the objects, but accurate estimates of the others.

Ehrenfeld [22] also suggested a definition of efficiency for experimental designs in general, which might as well be applied to Weighing Designs. The criterion suggested by him is based upon the maximization of λ_{min} (minimum eigen value of [X'X]).

In some situations, the above criteria would lead to an equivalent measure of efficiency.

It has been indicated in [14] that in defining the efficiency of a chemical balance design, the maximization of $|A|$ would be equivalent to the maximization of λ_{min}. However, in case of a spring balance design, the criterion of maximization of det.$|A|$

would be preferable in that this criterion would include the consideration of maximization of λ_{min}. Thus, in defining the efficiency of Weighing Designs, in general, a unified approach will be provided by Mood's "efficiency definition," that is, criterion (iii).

II. CHEMICAL BALANCE PROBLEM

1. Hotelling's Fundamental Lemma

Denoting by $A = [X'X] = (a_{ij})$, $i,j = 1,2,\ldots,p$, Hotelling [29] has proven the following Lemma.

If $a_{12},a_{13},\ldots,a_{1p}$ $(= a_{21},a_{31},\ldots,a_{p1}$ *respectively) are free to vary while the other elements of* A *remain fixed, the maximum value of* A/A_{11} *is* a_{11}, *and is attained when and only when* $a_{12} = a_{13} = \ldots = a_{1p} = 0$, *where* A_{11} *is the minor of* A *obtained by deleting the first row and column.*

The above Lemma says that the variance of $\hat{\beta}_1$, namely $\hat{\sigma}^2 A_{11}/A$, cannot be less than σ^2/a_{11}, and that the variance would reach this value, only if the experiment is so arranged that the elements after the first row and column of A are all zero. This minimum value, σ^2/a_{11}, will be attained, when the first column of X is orthogonal to all the others. It will also be clear that the *minimum minimorum* [29] of the variance will be reached, if the first column of X is not only orthogonal to all the others, but also if it consists entirely of +1's and -1's as its elements, so that $a_{11} = N$. N is the maximum possible value that a_{11} can take. The value of this *minimum minimorum* will thus be equal to σ^2/N.

The design matrix X in the scheme of weighings in (2) is an 8×7 matrix of +1 and -1. These seven columns are orthogonal. If a column of +1's is added to it, we get an 8×8 orthogonal matrix as shown below: (1 following the plus or the minus sign is not indicated.)

$$X = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & + & - & + & - \\ + & - & + & - & - & + & - & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \end{bmatrix}. \qquad (3)$$

The scheme (2) was designed to find the weights of seven light objects on a chemical balance, which was known to have a bias. If the bias is taken as an additional object whose weight is to be determined, and if the first column is made to correspond to the bias, the design matrix (3) (as shown above) would be suitable for estimating the weights of the seven objects and the bias, and would, in fact, be the best design for the purpose by virtue of Hotelling's Lemma [29].

Thus, the above scheme may be used as a design for finding the weights of eight different objects, if the balance is free of bias, or the weights of seven objects, if the balance has a bias.

The rows of matrix (3) would indicate how to combine objects in the weighing operations. The columns would refer to the objects

to be weighed. The matrix [X'X] will be a diagonal one, having 8 on its diagonal. The diagonal form of the matrix reduces the solution of the normal equations (pertaining to the least-squares estimates) to the trivial task of dividing by 8 each of the numericals occurring on the right hand side of the normal equations. That is to say, the estimates will be obtained as $\hat{\beta}$ = [X'Y]/8, as pointed out before.

In the above illustration, the inverse of the matrix [X'X] is also of the diagonal form, the elements in the diagonal being 1/8. The variance of each of the estimates will, therefore, be $\sigma^2/8$. With eight weighing operations it has been possible to obtain the least possible variance of $\sigma^2/8$ for each of the eight objects. If each object were weighed separately, 64 weighing operations would have been needed in all to arrive at this precision.

From the above, it will be evident that such a saving of weighing operations is possible not only in case of eight objects, but is possible of any number N of objects, provided an orthogonal matrix of order N of the above type with ±1 as its elements exists. Such matrices will serve as the chemical balance designs of maximum possible efficiency, as the *minimum minimorum* of the variance will be attained by each object.

2. Estimation of Error Variance

When scheme (3) is used for finding out the weights of eight objects from eight weighing operations, no degrees of freedom are left for estimating the error variance. If we have eight objects whose weights are to be determined, and if it is required that the

error variance be also estimated from the experiment, scheme (3) may be repeated twice so that the resultant design matrix be of dimensions 16×8. In such a case, 16-8 = 8 degrees of freedom will be left for estimating the error variance.

The same scheme may, however, be used for finding out the weights of objects less than eight in number. If, for instance, there are five objects whose weights are to be determined, any five columns of scheme (3) may be used as the Weighing Design. In that case, the matrix [X'X] will be of dimensions 5×5 and be diagonal with 8 on its diagonal. The variance of each of the five estimated weights will be $\sigma^2/8$, and 8-5 = 3 degrees of freedom will be left for estimating the error variance.

3. Chemical Balance Designs and Hadamard Matrices

From the details given above, it will have been evident that if N weighing operations are made to determine the weights of N objects, the minimum variance that each estimated weight might have had would be σ^2/N, and that this minimum variance would be reached when the design matrix X is orthogonal (orthogonal in the sense that [X'X] is diagonal) consisting entirely of +1's and -1's as its elements. Thus, the problem of finding the best chemical balance design is related, as pointed out by Mood [33], to Hadamard matrices and the Hadamard determinent problem [26].

We would denote such a matrix by H_N. If H_N exists for a given N, H_N would be the best chemical balance design for N=p. If the number of objects to be weighed is less than N, we may choose from H_N, a number of columns equal to the number of objects to be weighed.

Mood [33] referred to the work of Paley [34] and Williamson [42], and pointed out that H_{4k} existed for the range of

$$0 < 4k \leq 100,$$

with the possible exception of $4k = 92$.

At about the time Mood's paper [33] appeared, Plackett and Burman [35] also provided solutions of Hadamard matrices for all possible $N \leq 100$ except for $N = 4k = 92$. The Hadamard matrix for $4k = 92$ remained undecided till about 1962. The solution was first given by Baumert, Golomb, and Hall, Jr. [17]. (A comprehensive account of Hadamard matrices is now available in Marshall Hall, Jr. [27].)

4. Chemical Balance Weighing Designs when $N \neq p$

Kishen [31] pointed out that with $N = 2^m+1$ weighing operations, the most efficient chemical balance Weighing Design would probably be obtained by augmenting an H_N with a row, 1,.1,...,1. He also suggested that with $N = 2^m+r$ $(r<2^m)$ weighing operations, a highly efficient design would be available, if H_N is augmented with r rows of +1's. With reference to this latter observation of Kishen [31], Mood [33] pointed out that even with r=2, one would rather add two different rows of H_N to H_N than adding two rows of +1's.

The above observations clearly indicated the necessity for a detailed examination of the efficiencies of design matrices obtained by augmenting H_N with its rows which are not necessarily the same. A comprehensive account of such design matrices obtained with such augmentations is available in [4].

5. Chemical Balance Designs for Values of N=p When H_N Does Not Exist: Design Matrices When Det.|A| is Maximum

The following are some of the most efficient chemical balance designs furnished by Mood [33] for small values of N=p, when H_N does not exist. Here, a design has been called the most efficient (Mood's "efficient definition"), if the value of det.|A| = det.|X'X| is maximized, or - equivalently, if det.|C| is minimized. Mood [33] constructed these square matrices following a method due to Williamson [43]. (The interested reader may also refer to the work of Gordon [25] in this connection.)

There may be several designs with the same maximum value for det.|A|. When p=3, the best designs (omitting 1 following the sign) are

$$X = \begin{bmatrix} + & + & 0 \\ + & - & + \\ - & + & + \end{bmatrix}, \quad \begin{bmatrix} + & + & + \\ + & - & + \\ - & + & + \end{bmatrix} \text{ and } \begin{bmatrix} + & + & - \\ + & - & + \\ - & + & + \end{bmatrix}.$$

All of these matrices have det.|A| = 16 (which is considerably smaller than the value 27 that det.|A| would have, if an optimum design with an orthogonal matrix existed). The first of the above designs, for p=3, gives

$$\{c_{ii}\} = \{3/8,\ 3/8,\ 1/2\},$$

while the second and third give

$$\{c_{ii}\} = \{1/2,\ 1/2,\ 1/2\}.$$

For N=p=5, the two best designs are

$$X = \begin{bmatrix} + & + & + & + & - \\ + & + & + & - & + \\ + & + & - & + & + \\ + & - & + & + & + \\ - & + & + & + & + \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} + & - & - & - & - \\ + & + & + & - & - \\ + & - & + & - & + \\ + & - & + & + & - \\ + & + & - & + & + \end{bmatrix},$$

both of which have det.$|A| = 3^2 2^8$, and $\{c_{ii}\} = \{2/9, 2/9, 2/9, 2/9, 2/9\}$.

For N=p=6, the best design is

$$\begin{bmatrix} + & - & - & - & - & - \\ + & - & - & - & + & + \\ + & - & - & + & + & - \\ + & - & + & + & - & + \\ + & + & + & - & + & - \\ + & + & - & + & - & + \end{bmatrix},$$

which has det.$|A| = 5^2 2^{10}$, and $\{c_{ii}\} = \{1/5,\ldots,1/5\}$.

It is not known to the author, if any systematic attempt has been made to construct such square matrices with maximized determinant for larger values of N=p.

6. Other Designs of Comparable Efficiency

A method of construction of Hadamard matrices due to Plackett and Burman [35] was used in [5] to illustrate the construction of a 6×6 and a 10×10 orthogonal matrix consisting of +1, -1 and 0 as the

elements. These are reproduced below:

For N=p=6,

$$X = \begin{bmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & + & - \\ + & + & 0 & + & - & - \\ + & - & + & 0 & - & + \\ + & + & - & - & 0 & + \\ + & - & - & + & + & 0 \end{bmatrix}. \qquad (4)$$

For N=p=10,

$$X = \begin{bmatrix} 0 & + & + & + & + & + & + & + & + & + \\ + & 0 & + & - & + & - & + & - & + & - \\ + & + & 0 & + & - & + & + & - & - & - \\ + & - & + & 0 & - & + & - & - & + & + \\ + & + & - & - & 0 & + & - & + & + & - \\ + & - & + & + & + & 0 & - & + & - & - \\ + & + & + & - & - & - & 0 & + & - & + \\ + & - & - & - & + & + & + & 0 & - & + \\ + & + & - & + & + & - & - & - & 0 & + \\ + & - & - & + & - & - & + & + & + & 0 \end{bmatrix}. \qquad (5)$$

For (4), the variance factors are $c_{ii} = 1/5$. These are the same in magnitude as those under the 6×6 design matrix of maximized determinant given by Mood [33]. For (5), the variance factors are $c_{ii} = 1/9$. (It may be mentioned here that such orthogonal matrices may be constructed [35] for any N, where $N=p^n+1$, $p^n=4t+1$.)

Raghavarao [36-37] constructed two series of Weighing Design matrices of maximized efficiency for the cases when (i) N is odd, and when (ii) $N \equiv 2 \pmod 4$. In the construction of these matrices, a design was considered to be the best, when $1/\sum_{i=1}^{N} c_{ii}$ was the maximum , subject, however, to the restraints that

(1) the variances of the estimated weights be equal, and that

(2) the estimated weights be equally correlated.

7. Complexities Arising Out of Different Definitions of Efficiency: Comparison of Efficiencies

Different authors have chosen different criteria to define the efficiency of a Weighing Design. A Weighing Design which is considered to be the best on one criterion may not be considered so on the other. One might thus wish to compare these designs with reference to some of the desirable criteria, such as the magnitudes of the variance factors, the value of det.$|A|$, etc. Such a comparison is possible and has been presented in [15]. The discussion includes consideration of efficiencies also of rectangular matrices used as Weighing Designs.

III. SPRING BALANCE PROBLEM

1. Two Theorems of Mood Giving the Best Spring Balance Designs

As mentioned before, the spring balance problem is different from the chemical balance problem in that the elements x_{ij} of the design matrix X are restricted to assume values 0 and 1, whereas in the chemical balance problem, the elements of the design matrix can assume the values +1, -1 or 0.

For number of weighings N>p, Mood [33] has made the following approach to get the best design (best according to Mood's "efficiency definition") for a spring balance.

"Let P_r be a matrix whose rows are all the arrangements of r ones and (p-r) zeros, ($0 \leq r \leq p$). (The symbol should also have a subscript p, but that is omitted because any specific value for p will always be clear from the context.) The matrix will have p columns and $\binom{p}{r}$ rows. Let X be a matrix made up of matrices P_r arranged in vertical order. Let n_r be the number of times P_r is used in constructing X. X is a weighing design for p objects and $N = \sum_r n_r \binom{p}{r}$ weighing operations." Adopting these notations, Mood [33] has proven the following two theorems giving the best spring balance design.

Theorem (1): If p=2k-1, *where* k *is a positive integer, and if* N *contains the factor* $\binom{p}{k}$ *then* $|a_{ij}|(\det.|A|)$ *will be maximized when* $n_k = N/\binom{p}{k}$ *and all other* $n_r = 0$.

Theorem (2): *If* p=2k, *where* k *is a positive integer, and if* N *contains the factors* $\binom{p+1}{k+1}$, *then* $|a_{ij}|(\det.|A|)$ *will be maximized*

when $n_k = n_{k+1} = N/\binom{p+1}{k+1}$, *and all other* $n_r = 0$.

When p is odd, Mood [33] observed that P_k is a design which not only minimizes the confidence region (i.e., value of det.$|C|$) for estimating the weights, but also minimizes the individual variance factors. When, however, p is even, Mood [33] observed that the varaince factors may not be the minimum for the design suggested in Theorem (2). He, however, makes a surmise that the best design from the point of view of minimum variance factors would be made up largely from P_k and a small proportion from P_{k+1}.

2. Spring Balance Weighing Designs and Balanced Incomplete Blocks:

Theorems of Mood [33] imply that if, for instance, the number of objects is 15, then the total number of weighing operations has to be as large as $\binom{15}{8}$ in order that the maximum possible precision be reached. This would obviously require a very large number of weighing operations. Even for a smaller number of objects, say 7, we would need to make $\binom{7}{4} = 35$ weighing operations in order to secure the maximum possible efficiency.

However, it was pointed out in [1] that spring balance designs of equivalent efficiency but based on a smaller number of weighing operations would be available from the arrangements given by Balanced Incomplete Block Designs (BIBD) discussed in Fisher and Yates [24] and in Bose [19].

In weighing designs, v of a BIBD will take the plaee of p, the number of objects to be weighed, and b that of N, the number of weighings that can be made. The matrix [X'X] in this case will take

the form of a matrix with r on the diagonal and λ elsewhere. The variance factor for each estimated weight [1], as mentioned before, is

$$\frac{r + \lambda(p-2)}{(r-\lambda)\{r+\lambda(p-1)\}},$$

where p is the number of objects to be weighed, and r and λ have the same meanings as understood in the theory of BIBD's, that is, r is the number of times each object is weighed, and λ the number of times each pair of objects is weighed together.

Though the *minimum minimorum* of σ^2/N can never be attained in a spring balance design, σ^2/N may still be held as the standard with reference to which the efficiency of a given design may be compared. The efficiency of the above design will then be obtained [1] as

$$\frac{(r-\lambda)\{r+\lambda(p-1)\}}{N\{r+\lambda(p-2)\}}. \qquad (5)$$

Replacing N by b, and p by v to accord with the notation of weighing designs, the two identities (as known in the theory of BIBD's) may be rewritten as

$$r = Nk/p, \qquad \lambda = r(k-1)/(p-1).$$

Substituting these values in (5), we obtain the efficiency factor in the form,

$$\frac{k^2(p-k)}{p(pk-2k+1)}, \qquad (6)$$

where k is the number of plots per block, or the number of objects that can be weighed at a time.

If, instead of adopting repetitions of P_k, only $\binom{p}{k}$ weighings were made in all, the efficiency factor for such designs (that is, designs with all possible combinations of k objects at a time) would be obtained as

$$\frac{(r-\lambda)\{r+\lambda(v-1)\}}{b\{r+\lambda(v-2)\}},$$

where $r = \binom{v-1}{k-1}$, $\lambda = \binom{v-2}{k-2}$, and $b = \binom{v}{k}$.

The above expression on simplification reduces to (6). Thus, the efficiency of a BIBD as a Weighing Design is the same as that of the designs, P_k, of Mood [33].

3. Series of Designs Denoted L_N by Mood

Mood [33] has also constructed spring balance designs of maximum efficiency, when N=p and N≡3 (mod 4). He has shown that in such a case the spring balance design of maximum possible efficiency is given by H_{N+1}, if it exists. For ready reference; the method of construction as given by Mood [33] is reproduced below:

Let K_{N+1} denote the matrix formed from H_{N+1} by adding or subtracting the elements of the first row of H_{N+1} from the corresponding elements of the other rows in such a way as to make the first element of each of the remaining rows 0. Obviously,

$$\left|K_{N+1}\right| = \pm\left|H_{N+1}\right|.$$

Excepting the first row, the elements of K_{N+1} are 0 and ±2, with signs of the non-zero elements being the same for elements in

the same row. Let L_N be the matrix obtained by omitting the first row and column of K_{N+1}, by changing all non-zero elements to +1, and by permuting two rows, if necessary, to make the determinant of L_N positive. Then,

$$\left|H_{N+1}\right| = 2^N \left|L_N\right|.$$

It will be clear from the above that, given L_N, one could reverse the procedure and determine an H_{N+1}. Similarly, there is a correspondence, in general, between square matrices with elements ± 1, and square matrices of one order less with elements, 0 and 1. The ratio of the values of the corresponding determinants is always 2^N, if their determinants do not vanish. Hence,the (0,1)-determinant will have its maximum possible value when its corresponding (± 1)-determinant has the maximum possible value. Thus, det.$|L_N|$ has the maximum value.

4. Connection of L_N with Symmetrical BIBD of a Special Kind

An 8×8 orthogonal matrix (matrix (3) for instance) would afford an L_7 as given by

$$L_7 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The above L_7 is the same as L_7 of Mood [33] with a rearrangement of its rows. L_7 is easily recognized, as pointed out in [1], to be a symmetrical BIBD for v=b=7, r=k=4, and λ=2. Such a design is characterized by an *orthogonal series* as discussed in [24].

The distribution of 1's in the above scheme would show in what combination the objects have to be weighed.

In an L_N, not only det.$|A|$ is maximized, but the variance factors are also minimized, as pointed out by Mood [33]. It has, however, been shown in [3] that there are no combinations of the values of the parameters in a BIBD other than those in an L_N, for which the variance factors reach the minimum value of $4N/(N+1)^2$.

5. Use of BIBD's as Weighing Designs

When a BIBD is used as a Weighing Design, the solutions of the normal equations giving the estimated weights (least squares) can be reduced to a simple routine [9].

The estimated weights are obtained as

$$\hat{\beta}_i = \frac{1}{(r-\lambda)}\left[Z_i - \frac{\lambda T}{r+\lambda(v-1)}\right]$$

$$= \frac{1}{(r-\lambda)}\left[Z_i - \lambda t/r\right],$$

where Z_i $(i = 1,2,\ldots,p)$ are the elements of the vector X'Y, $T = \sum_{i=1}^{v} Z_i$, $t = \sum_{i=1}^{b} y_i$, v=p, and b=N.

In particular, when L_7 is used as the design, the estimates are obtained as

$$\hat{\beta}_i = \frac{1}{2}(Z_i - t/2).$$

6. Efficient Spring Balance Weighing Designs for Small Values of N=p

Mood [33] quoted from Williamson [42] some square matrices with maximum value for det.$|A|$ to furnish weighing designs of maximum possible efficiency for small values of N=p. These square matrices are denoted as D_p, p showing the dimensions. For example, we have for N=p=4,

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The value of the det.$|A| = 9$. The variance factors are all 7/9 for the first, and for the second, $c_{ii} = \{7/9, 7/9, 4/9\}$. For N=p=5, we have

$$D_5 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The value of the det.$|A| = 25$, and the variance factors are $c_{ii} = \{19/25, 19/25, 16/25, 11/25, 16/25\}$. For N=p=6, we have

$$D_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The value of det.$|A|$ = 81, and the variance factors are

c_{ii} = {17/27, 17/27, 17/27, 17/27, 17/27, 17/27}.

It is not known to the author, if such square matrices have been constructed for higher dimensions in general.

7. Spring Balance Designs when Different Rows of L_N are Added to it

We have seen that H_N is the best chemical balance design for N=p when it exists, and that L_N is the best spring balance design for N=p when H_{N+1} exists.

It has been indicated in [4] that efficient chemical balance designs could be constructed between the dimensions of H_N and H_{N+4} by the addition of different rows of H_N to H_N, and that some of these rectangular design matrices are comparable to chemical balance designs which were otherwise known to be the most efficient. A detailed discussion in this regard has been furnished in [15].

In case of a spring balance also, similar rectangular design matrices may be constructed between the dimensions of L_N and L_{N+4} by the addition of different rows of an L_N to the L_N. A detailed discussion of such augmentations has been presented in [3].

8. Construction of Weighing Designs by the Inclusion of Additional Row to a Design Matrix

It has already been pointed out that it is possible to get the estimates of N weights with N weighings in a chemical balance with maximum efficiency, if the Hadamard matrix H_N (N=4k) exists. If the Hadamard matrix of the next higher order (that is H_{N+4}) is

available, it would be possible to estimate the weights of N+4 objects with N+4 weighings. No general solution, as far as known to the author, is yet available by which it is possible to get chemical balance design of maximum efficiency to estimate the weights of (N+m) (m = 1,2,3) objects with (N+m) weighings, when N=4K.

It has been indicated earlier that a spring balance design of maximum efficiency to weigh (N-1) objects with (N-1) weighings is available, if Hadamard matrix H_N exists. Here also, no general solution appears to have been attempted for the construction of a square matrix between the two most efficient spring balance designs of successive orders.

Williamson [42-44], however, has outlined for small values of N, a method of constructing determinants of order N+m (m = 1,2,3) with maximum possible values. The corresponding matrices may be utilized as the most efficient weighing designs. (Most efficient as per Mood's "efficiency definition.")

In the absence of the above matrices for higher dimensions, there is some interest in knowing the most efficient rectangular design matrices of order $(N+m)\times N$ (m = 1,2,3), so that it might be possible to estimate the weights of N objects with N+m (m = 1,2,3) weighings with the maximum possible efficiency. A procedure of constructing such rectangular matrices has been outlined in [7]. This was done by way of indicating a result which may be called some sort of an extension of a result by Williamson [42-44].

9. Use of Partially Balanced Incomplete Block Designs as Weighing Designs

We have seen that arrangements given by BIBD's may be used as efficient spring balance designs. The most efficient spring balance weighing designs, L_N, of Mood [33] are, as indicated earlier, afforded by a special class of symmetrical BIBD's.

It has been shown in [10] that a general class of combinatorial arrangements, developed by Bose and Nair [20], known as Partially Balanced Incomplete Block Designs (PBIBD), may also be used as Weighing Designs in some cases. But, all such PBIBD's cannot be used as Weighing Designs for estimating the weight of each object uniquely in the least squares sense, as the design matrix X might be singular in some cases. An indication has been given in [10] under what conditions a design matrix coming from a PBIBD may become deficient in rank.

Incidentally, the D_6 quoted by Mood [33] for a spring balance Weighing Design for N=p=6, maximizing det.$|A|$ may easily be recognized [10] to be a PBIBD with the following parameters, v=b=6, r=k=3, λ_1=1, n_1=4, λ_2=2, n_2=1,

$$p_{ij}^1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad p_{ij}^2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

IV. MISCELLANEOUS ISSUES CONCERNING THE WEIGHING PROBLEM

1. On the Determination of Total Weight:

An Optimum Chemical Balance Design is Not the Best Design for the Estimation of Total Weight

An orthogonal design which has the maximum efficiency in determining individually the weights of p objects in a chemical balance is not the best design for the estimation of a linear function of the objects. To illustrate this, let there be three objects, β_1, β_2, β_3, the weights of which have to be determined on a balance corrected for bias, and let us, for this purpose, adopt the following design,

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}. \qquad (7)$$

In the above design, the variance of each of the estimated weights is $\sigma^2/4$, which is the *minimum minimorum*, and, as such, the design has the maximum efficiency in the estimation of the weights of the individual objects. But, in estimating a linear function of the objects, for instance, the total weight, designs more efficient than this are available.

The variance of $\ell_1\hat{\beta}_1 + \ell_2\hat{\beta}_2 + \ell_3\hat{\beta}_3$ is known to be equal to

$$\sum_{i,j=1}^{3} \ell_i \ell_j c_{ij} \sigma^2,$$

where c_{ij} denotes the elements of $[X'X]^{-1}$. As the above design furnishes the estimates orthogonally, variance of the estimated total weight will be given by $3\ \sigma^2/4$.

If, instead, the design given by the matrix,

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (8)$$

is adopted, the variance of the estimated total weight may be easily seen to be $3\sigma^2/7$. $3\sigma^2/7$ is less than $3\sigma^2/4$. Therefore, with four weighing operations, the design given by [8] is more efficient in estimating the total weight than the design given by [7]. A still more efficient design for estimating the total weight is simply to weigh all the objects together four times. The necessity for an efficient design to estimate any linear function of the objects (or the total weight) will perhaps arise only when the objects cannot all be weighed at a time collectively on a single pan.

In the estimation of total weight, efficient design is also afforded by the arrangements of a BIBD, as the covariances are all negative. It has been shown in [1], that the variance of the estimated total weight, using a BIBD, reduces to

$$\frac{p\sigma^2}{r + \lambda(p-1)}.$$

In actual practice, an object may break into pieces, and we may need to know their total weight (see Rao [39]).

Questions concerning the construction of weighing designs which will be good both for "individual" estimation and "total" estimation were considered in [7].

2. Factorial Approach to the Weighing Problem: Fractional Replicates as Weighing Designs:

Kempthorne [30] discussed the weighing problem from the point of view of "factorial experimentation." He suggested that the problem of weighing a number of objects be rather regarded as the problem of estimating the effects of a number of factors which do not interact. The motivation in this suggestion appears to be in keeping with the tenor of the original example of Yates [45] which was, in fact, meant to show that one could think of a factorial experiment where factors do not interact.

In this paper [30], Kemptorne gave a brief description of the method of constructing fractional replicates of a 2^n-fractorial experiment, originally discussed by Finney [23], and gave an illustration showing how a fractional replicate could be used as a weighing design to find the weights of ten different objects, a, b, c, d, e, f, g, h, k, l.

Kempthorne [30] mentions further that the precision can be increased four fold by interpreting the absence of each letter as the placing of the object in the other pan, in case a chemical balance could be used. This improvement is, again, of the same nature as indicated by Hotelling [29] with reference to the original illustration furnished by Yates [45].

Kempthorne [30] mentions that the fractional replicate designs have following useful properties:

(i) the design automatically takes care of any bias in the balance;

(ii) the effects or the weights may be computed easily as indicated above;

(iii) the effects are uncorrelated;

(iv) all the effects are measured with the same precision, and

(v) an estimate of the experimental error which is independent of the effects may be computed from the results.

3. Kempthorne's Observations About the Non-orthogonality of the Estimates Furnished by L_N of Mood

Of the above properties, property (iii) is of special significance. While referring to the spring balance designs, L_N, of Mood [33], Kempthorne [30] mentioned that although these optimum designs of Mood furnish "somewhat" smaller variance than what is given by the *fractional replicates*, these designs (the optimum designs of Mood) have the disadvantage that the estimates are correlated, whereas estimates furnished by fractional replicates are orthogonal.

It has, however, been shown in [2] that the designs given by fractional replicates suggested by Kempthorne [30] are virtually the same as the designs, L_N, of Mood. The designs furnished by fractional replicates take account of the bias, and if the weighing operation corresponding to the determination of the bias is omitted

(in case the spring balance is free from bias), the resultant design matrix will be the same as L_N of Mood, and will, therefore, fail to give orthogonal estimates. Again, these optimum designs (L_N of Mood) may also be made to furnish orthogonal estimates when these designs are adjusted to suit estimation in a biased balance. For example, it is true that the design matrix, L_3, given by

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

does not give orthogonal estimates as such. But, let us assume that the spring balance has a bias, and that it is required to find the estimates of weights free of bias. It would then be necessary to modify the design matrix to suit the required estimation. The modification would require that one more column be added to the design matrix X to correspond to the estimation of the bias. Taking the first column to correspond to the bias, and making an additional weighing operation on empty pan to determine the bias, we shall have the design matrix modified as

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

In terms of factorial experiments, the above design could be regarded as a 1/2 replicate of a 2^3-factorial experiment being given by the identity relationship I = ABC. The combinations are (1, ab, ac, and bc).

The first row of the above design represents a weighing operation on "empty pan," and would thus go to the determination of the bias. The remaining three weighing operations would represent weighings of the three objects, taken two at a time along with the bias. This does not mean, however, that we need to make any fresh weighing operations. We would, in fact, use the same records of weights which were available on the supposition that the balance had no bias. Only one additional weighing is needed, and this is a weighing on empty pan. In substance, therefore, this modification really means that we need to make only one additional weighing operation, and that, on empty pan.

When the modification is made as above, we would be able to determine the weights of the objects free of bias, and in mutually orthogonal linear combinations of the observations.

For the modified design matrix, as given above, $[X'X]^{-1}$ will take the following form [2]:

$$[X'X]^{-1} = \begin{bmatrix} 1 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1 \end{bmatrix}.$$

It will be noticed from the above that the variance of an estimated weight is σ^2, whereas the same for an L_3 is $3\sigma^2/4$.

All the rows of $[X'X]^{-1}X'$, except the first, are orthogonal to one another meaning thereby that the estimates given by $\hat{\beta} = [X'X]^{-1}X'Y$ (except that for the bias) are linear functions of the y's which are mutually orthogonal. In fact, for this design, $[X'X]^{-1}X'$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

The last three rows of the matrix show that the three estimates are obtained as orthogonal linear functions of the observations.

The above property will hold good for any N of L_N. As L_N furnishes a special kind of a symmetrical BIBD, it would be natural to ask if this property of orthogonality of the estimates may be made to extend also to BIBD's in general. In fact, it would, and this topic has been discussed in [8].

The distribution of 0's in L_N gives the complementary BIBD for which $r_0 = r-1$, $k_0 = k-1$, and $\lambda_0 = \lambda-1$. If, to such a complementary design, a row of ones and a column of ones (in that order) be added to suit estimation in a biased spring balance, exactly a similar situation will be obtained [2], in that the estimates will be mutually orthogonal. The variance factors of the estimated weights will remain the same [8] as in the case of augmentation of the original design L_N. It can be readily verified that the original design of Yates [45] to determine the weights of seven light objects and a bias is an illustration of this kind. If a column of ones be added to the design of Yates [45], the resultant scheme will be an 8×8 design matrix, and this design matrix will be what may be obtained by the addition of an additional row of ones and a column of ones in that order to the design complementary to L_7.

4. Use of Fractional Replicates of Other Types

While referring to the construction of optimum designs for chemical balance through fractional replicates, Kempthorne [30] mentioned about the possibility of using a 3/4 replicate as a Weighing Design. It has been shown in [2] that, when a 3/4 replicate of a 2^n-factorial experiment is used as a chemical balance weighing design, the variance factors come out as $1/2^{n-1}$. Again, the same, $1/2^{n-1}$, is the variance factor, if the fractional replicate is of the type $(2^\beta-1)/2^\beta$, $(1<\beta<n)$. A detailed discussion in this regard has been presented in [2,3].

5. Weighing Designs Under Auto-Correlation of Errors

We have seen in the preceding pages that there are some weighing designs which have the maximum possible efficiency. These designs were efficient within the frame of the model where the error structure was assumed to be of the form, $E(\varepsilon\varepsilon') = \sigma^2 I_N$. It may perhaps be desirable to examine how the efficiencies of those designs would alter, if at all, if the errors are assumed to be autocorrelated, that is, if the error structure is assumed to take the form $E(\varepsilon\varepsilon') = \sigma^2 V$. Some results in this direction have been indicated in [11].

6. Singular Weighing Designs: Nature of the Problem

The design matrix X has so far been assumed to be of full rank (rank p). When X is of full rank, the matrix [X'X] is non-singular. But, it is possible that, as a result of "bad designing,"

or despite the best of intentions, the chosen design matrix X has rank less than the full. Consequently, the matrix [X'X] will be singular. Thus, one may have to deal with the problem of what may be called a Singular Weighing Design.

It is known that when X is not of full rank, it is not possible to have a unique (Unbiased) estimate of each of the objects under non-randomized procedures (see Zacks [46]). It would be possible, however, to have a unique unbiased estimate of an estimable linear function of the objects. Thus, in a Singular Weighing Design, it may be required to ascertain if it would be possible to furnish an estimate of a given linear function of the objects (say, "total weight").

If, after taking observations in accordance with a design, it is detected that the design matrix is singular, we may be required to take additional weighings to make up for the deficiency in rank. In that case, the problem would be to determine how best we could take up this augmentation procedure. Results in this direction have been discussed by Raghavarao [38] and in [12].

7. Fractional Weighing Designs:

Nature of Fractional Weighing Designs

It has been mentioned earlier that for the estimates of weights of individual objects, the best design for a chemical balance is given by a Hadamard matrix X of dimensions $p \times p$, when it exists. If, for lack of resources, time or for other reasons, only r rows of X are to be used for the weighing operations, the resultant

design matrix will be of dimensions r×p and will be a *fraction* of the full design matrix X. The resultant matrix will necessarily be singular. While it is not possible with such a (*non-randomized*) singular, fractional weighing design to provide unique and unbiased weights of the individual objects, it may be possible to find, as it is well known, unique unbiased estimates of some linear functions (estimable) of the objects. But, fractional weighing designs under "randomized procedures" would provide unbiased estimates for any linear function $\lambda'\beta$ (which, of course, would include any component of β).

Zacks [46] visualized the possible use of such randomized, fractional weighing designs. The fractional weighing design is obtained by choosing at random n($n<p$) rows, independently and with replacement, from a given Hadamard matrix according to a probability vector $\underline{\xi}$ of order p. Each of these n rows would specify a weighing operation to be performed.

A randomization procedure has also been developed by Zacks [46] which would afford unbiased estimation for any $\lambda'\beta$ with minimum variance. The probability vector for the randomization procedure would, of course, depend upon the linear functional λ. It has, in fact, been indicated by Zacks [46] that every functional λ specifies a subset of, say r($1\leq r\leq p$), admissible weighing operations (rows of the Hadamard matrix) in the sense that if other weighing operations are chosen, the estimation procedure would either be biased, or would have a variance larger than the one obtainable under admissible weighing operations. If each of the r admissible weighing operations

($1 \leq r \leq p$) is chosen with probability equal to $1/r$, the corresponding unbiased estimator will have a uniformly (in β and σ^2) minimum variance.

Finally, the formulae of the unbiased estimator of $\lambda'\beta$ and its variance have been extended to the case of random choice, without replacement, of n rows out of the r admissible ones.

8. Some Analogous Results Obtained Under Non-randomized Procedure

Some results of connected interest have been indicated in [13] with respect to fractional weighing designs which, of course, are obtained without resorting to any randomization procedure. It has been shown [13] what connection the linear functional λ has with the fractional weighing design and, in that context, it has been pointed out to what extent we could be arbitrary in the selection of the components of λ. The structure of the estimable linear function along with the variance of its estimate has been spelled out in full, bringing out the connection of this variance with the variance as obtainable under the full design matrix. It has been shown that with a fraction, which, of course, would depend on λ, we could obtain the *same precision* for the estimate as would be obtainable in a full design matrix without, of course, having to perform all the weighing operations. This fraction is, in fact, the same as given by the set of "admissible rows," referred to by Zacks [46] under *randomization procedure*.

9. Extension of the Procedure to the Spring Balance Problem

Beckman [18] has extended the results obtained by Zacks [46] to the spring balance problem. In the development of the procedure, he characterized a design to be the best, if the trace of the covariance matrix of the estimator was the minimum. In [18], both *randomized* procedures and *non-randomized* analogs have been studied. The results are similar to those obtained in [46] by Zacks and in [13] in case of the chemical balance problem.

10. Biased Estimation in Weighing Designs:

Problem of Biased Estimation in Regression

The model for Weighing Designs is also the model for multiple regression in general. In regression problems, it sometimes happens that the matrix of normal equations is "ill-conditioned" with the consequence that it becomes difficult to solve the system of normal equations. Such a difficulty is, in some situations, circumvented by the addition of a small positive quantity k to the diagonal elements of the matrix [X'X]. When such a small quantity is added, the normal equations take the form $[X'X+kI]\ \hat{\beta}^* = X'Y$. The estimator $\hat{\beta}^*$ obtained from solving these equations will obviously be biased.

Hoerl and Kennard [28] have studied a procedure of such biased estimation in the regression problem in general, and have indicated how the value of k could be chosen so that the total mean square error of $\hat{\beta}^*$ be less than the total variance of the least squares unbiased estimator $\hat{\beta}$.

Sihota and Banerjee [40] have studied the effect of such biased estimation procedure in Weighing Designs, where the design matrices are of a special nature, the elements x_{ij} of X being either (±1,0) in the chemical balance, or (1,0) in the spring balance.

Biased estimates have been compared in [40] with the unbiased estimates along with a corresponding comparison of the mean square error and the variance.

11. Repeated Spring Balance Weighing Designs

When a symmetrical BIBD is used as a spring balance weighing design, no degrees of freedom are left for the estimation of the error variance, as for this design we have N=p. In such a situation, the symmetrical BIBD could perhaps be repeated in order to secure "degrees of freedom" for the estimation of the error variance. But, Dey [20] has pointed out that if in a symmetrical BIBD, b>2r, then one should combine the symmetrical BIBD with its complementary BIBD rather than repeating it. Again, to meet the same requirement, Kulshreshtha and Dey [32] have suggested yet one more alternative design which would be preferred to "repeated designs" and to those suggested by Dey [21], provided one is interested in estimating the weights of some of the objects with increased precision at the cost of precision for others. However, it has been demonstrated in [16] that with a given problem on hand, one may repeat the complementary BIBD, and that this repetition would be preferred to the designs suggested by Dey [21] and Kulshreshtha and Dey [32]. This conclusion has been supported in [16] by a critical analysis of efficiencies of various alternative designs in this category.

12. Concluding Remarks

Finally, it might perhaps be appropriate to make one or two concluding remarks on the future of the weighing problem, which, though discussed in the language of weighing operations, would be applicable, as pointed out by Mood [33], to any problem of measurements, where the measure of a combination is a known linear function of the separate measures with numerically equal coefficients.

It will have been evident from a perusal of the preceding pages that there is a need for chemical and spring balance designs of square and rectangular matrices of higher dimensions such that det.$|X'X|$ is maximized, or trace of $[X'X]^{-1}$ is minimized without any restriction on the equality or otherwise of the variances and covariances. It is hoped, some progress will be made in this direction in the near future.

Again, the scope of the problem may perhaps be broadened further as and when other side issues are resolved to meet *ad-hoc* needs. For example, one might like to know the form the error structure would take in the situation when the "objects" are not *necessarily light*, and knowing the error structure, one might like to know how one should proceed in the selection of the best design.

On the other hand, however, one might pertinently ask how far these Weighing Designs have been found, or may be found, useful in practice. Professor Hotelling once told the author of this article that he knows of some chemists who had been utilizing these designs in their day-to-day routine. Possibly, many others are also utilizing Weighing Designs as such. Perhaps, some are making

use of the principles involved. As regards making use of the "principles," it may be mentioned that this author himself once used them in an estimation problem in a socio-economic survey. He hopes, however, that these designs will eventually attract a wider attention of statisticians, and that these designs will in future find their rightful place in "laboratory work" where weighing operations are needed as a matter of routine.

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Addendum

As pointed out in section 6 of chapter II, Raghavarao [36-37] constructed two series of efficient Chemical Balance Weighing Designs, when N is odd and when $N \equiv 2 \pmod 4$, subject to restraints that the variances and covariances of the estimated weights be the same. In a construction procedure, Raghavarao adopted a method due to Williamson [42]. Bhaskararao also constructed some similar weighing design matrices when N is odd, while matrices for $N \equiv 2 \pmod 4$ have been constructed by Ehlich, Ehlich and Zeller, and Yang. These references are listed below.

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EXPERIMENTAL TESTING OF INTRUSION DETECTION DEVICES

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ABSTRACT

This paper describes the planning factors and conduction of a test program designed to evaluate a relatively large number of line intrusion detection devices in an economic manner. The program intailed determination of probability of detection for a range of threat situations and the false alarm rates for a number of sensor systems.

Some typical results are presented to reflect the relevancy of the plans to the end results.

INTRODUCTION

Currently, a series of experiments are being conducted to test and evaluate various types of intrusion detection devices with respect to their probability of detection and false alarm rates. The sensors tested fall into the following detection categories.

1. Magnetic
2. Seismic
3. Electrostatics
4. Balance Pressure System

This paper will discuss the overall structure of the testing program and present generalized results from the experimentation. Due to security requirements, the exact name of the individual sensor cannot be mentioned only its class is mentioned.

SENSOR DETECTION TEST CATAGORIES

TEST PLAN TYPES	TEST PLAN MAKE UP	CLOTHING	TYPE TRAVEL	TARGET ROUTINE
BY AREA	• QUIET PERIOD BEFORE TRIALS	MAG. CLEAN	SLOW WALK	ROAD
BY SENSORS (LINE POINT)	• TYPE & NUMBER OF TARGETS	NORMAL	MED. WALK	TRAIL
BY ALL AREAS	• VEHICLE	W/WEAPON (OVERHEAD)	FAST	a. ALL STAKES
MOVING TARGET	• PEOPLE		CRAWL	b. SELECTED STAKES
FALSE ALARM	• COMBINATION		HOP	c. AREAS WITHIN STAKES
BAD WEATHER	• AIRCRAFT		JUMP	
NORMAL			RUN	PERPENDICULAR TO LINE
QUICKIE				

Figure 1

INTRUSION & FALSE ALARM STIMULI

Intrusion Stimuli

Sensor evaluation has historically been based on measurement of the ability to detect against a suspected or known threat and the capability to suppress false alarms from stimuli or from nuisance alarms. Different detection tests were required for each type of intrusion device tested in this experiment. The detection tests selected were expected to provide data for a sufficiently broad range of threats, so that sensor performance could be realistically evaluated and sensor selection might be made by a user for a given threat condition. Figure 1 is a matrix of all possible intrusion modes and intruder configurations for the various detection systems tested. Selection of the intruder characteristics were chosen from this test matrix for the various sensors tested. Test matrix entries of intruder/intrusion characteristics are chosen to explore the full range of the sensor's performance under an actual/simulated intrusion.

False Alarm Stimuli

False alarms generally are all alarms not associated with actual intrusions. Since these alarms would necessitate the same response by protective forces as an actual intrusion, their occurrence is indesirable and their suppression mandatory. One of the objectives of the testing was to assess the false alarm frequency of each system being evaluated, determine sources and relative effects, and recommend possible means of discriminating against the false alarms. False alarms were categorized by source/type:

1. False alarms - caused within the sensing system; defective electronics, unbalance (i.e., in pressure lines).
2. Nuisance alarms - caused by animals, aircraft, vehicle or storm activating a properly operating sensor in a manner similar to that of an intruder.
3. Unknown alarms - An alarm caused by unknown physical stimuli (shifts in earth crust, distant lightening).

Since the intent of the false alarm testing was to classify the false alarm sources, the test range, as shown in figure 2, was placed under tight control and surveillance to minimize local alarm sources and observe all, if possible, sources of alarms. Tests were run both day and night; flood lights in areas A & B provided adequate visibility for the tower observer and hand held spotlights were used for the other areas.

The local sources of the nuisance alarms generally were:

- Aircraft (jets, propeller aircraft and helicopters).

These were normally random, there sources being from local Army fields and civilian airports in the area.

- Trains

An active line passed a mile from the edge of the test range.

- Vehicles

Normally all roads entering and exiting the test area were blocked to eliminate the vehicle source. However, in some instances it was necessary to allow security vehicles and others into the area during testing.

- Animals

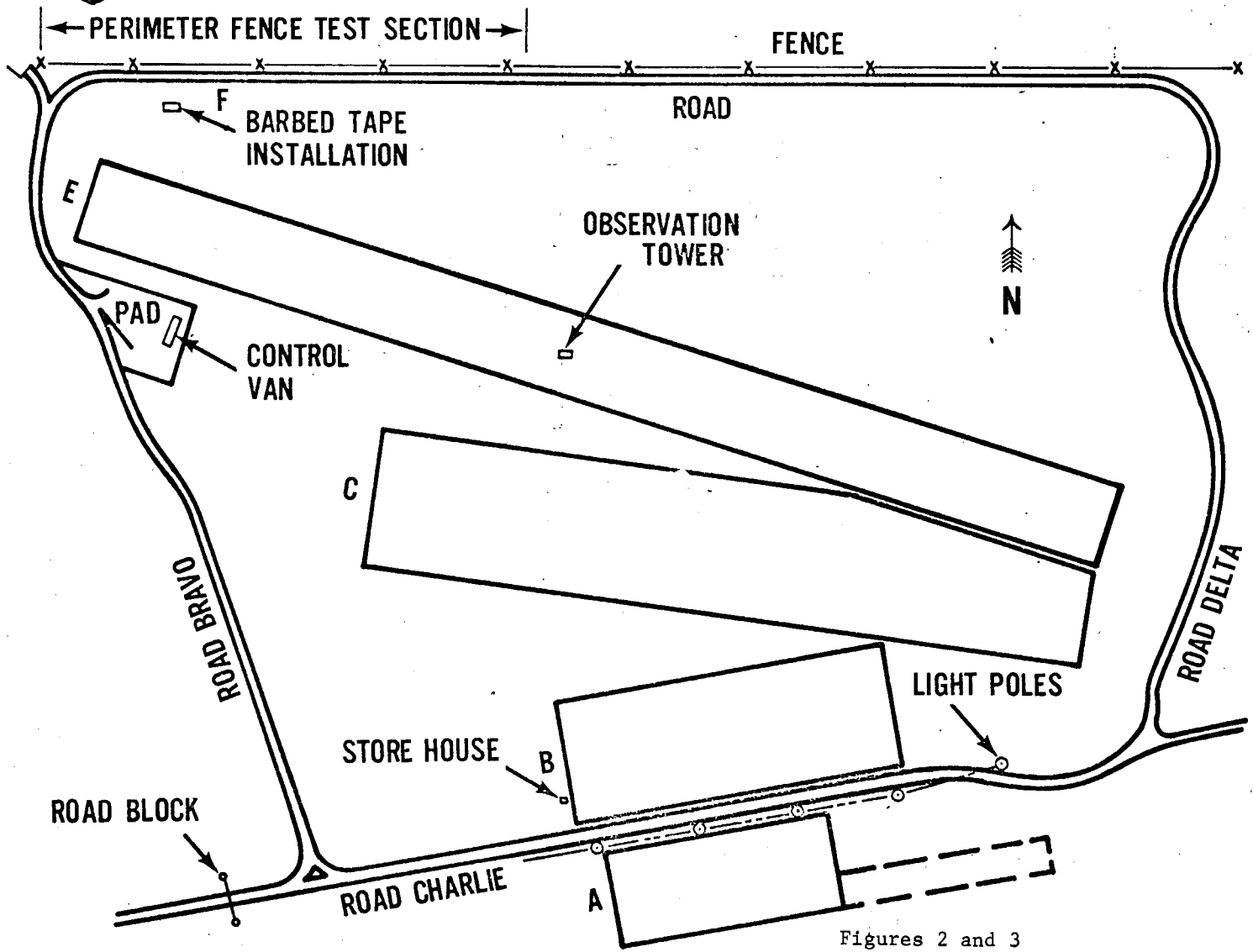
Animals frequently entered the test range area. The most prevalent and easiest to identify were elements of a herd of deer. Ground

hogs, possum, rabbits, squirrels and birds were also identified as alarm sources.

- Unidentified sources

Although every effort was made to identify alarms, the large area to be covered and the number of sensors precluded 100% visual coverage. These alarms were placed in the unclassified categories.

WOODBRIDGE TEST FACILITY



Figures 2 and 3

TEST PLANS AND PROCEDURES

Field Sensors

The test range layout, as shown in Figure 3 was planned to place the sensors as close together as possible in the designated ranges by length categories to conserve pass time. The sensor layout and intrusion paths were designated to provide stimulus for as many sensors as possible in each pass. This was also a consideration in planning the perimeter fence tests. But here, the engagement of both line and point type sensors in a comprehensive and economic fashion, was a factor. Planning requirements were, to define an adequate number of passes for each type of sensor, intruder configuration and intrusion mode to provide a result with the highest statistical confidence obtainable within the program time constraints.

For the line type sensor, up and back passes, at each interval point along the sensor were planned for each configuration and mode, which in the case of the magnetic sensors with three intruder configurations necessitated additional intrusion pass sets.

It was planned that each sensor be engaged every 10 meters regardless of length, however, this interval was modified to every 30 meters for the longer sensors because of the time involved. Where data analysis found anomalies at these 30 meter points, additional runs were scheduled every 10 meters along the entire length of the sensor.

Figure 4 shows a typical test path of an intruder subject along the stake paths over the several sensors in range A.

TYPICAL LINE SENSOR INTRUSION TEST PATTERN

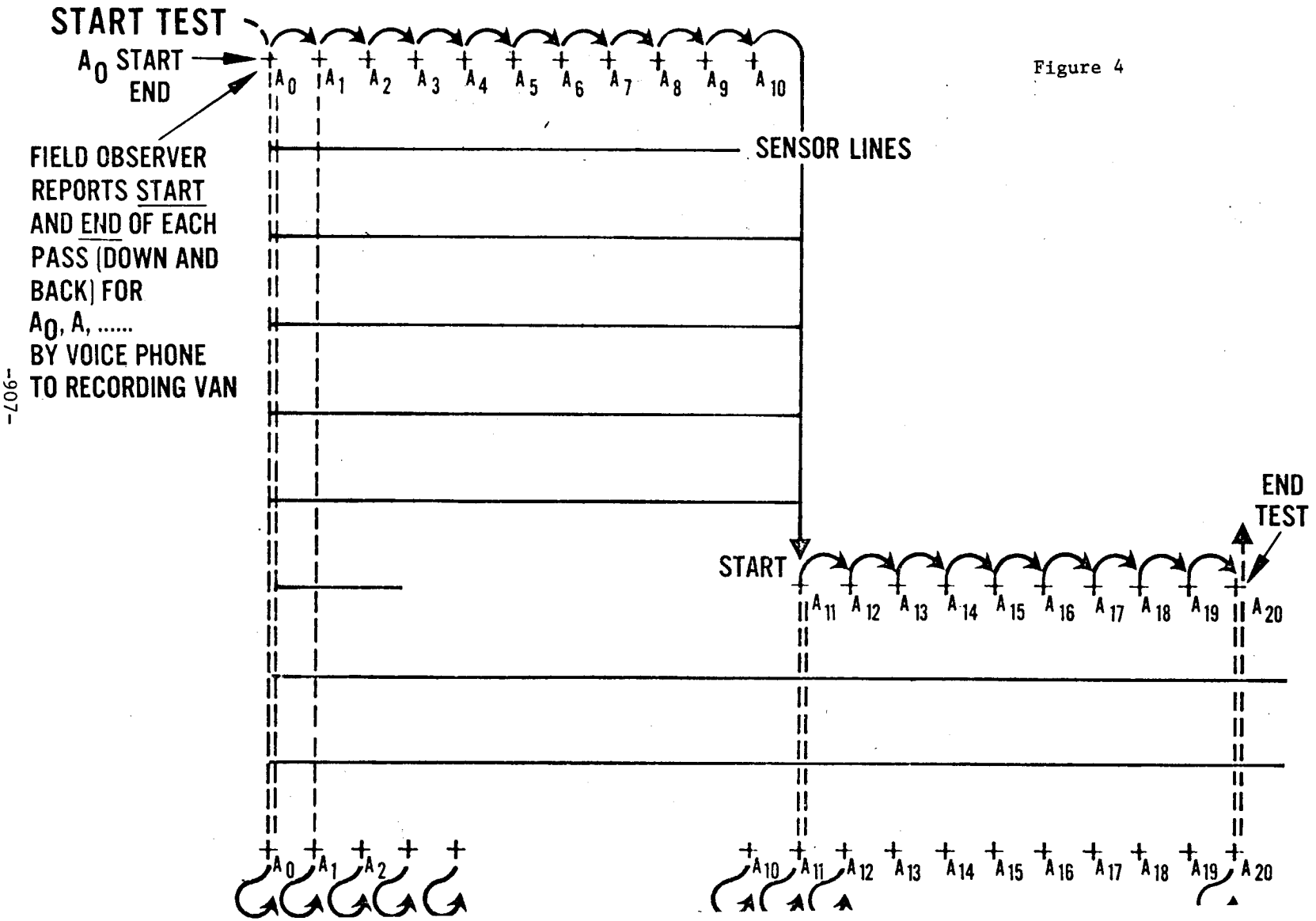


Figure 4

False Alarm Testing

The Program Test Plan provided for rigid range, vehicular and personnel traffic control, careful sensor output momitoring and recording, and magnetic tape recording of seismic, magnetic and acoustic background levels and for special false alarm source observation capability. The test plans and procedures required simple checkouts of each active sensor at the beginning and end of each test, and periodically (every 4 hours) during a false alarm test.

The test procedures principally specified the data recording requirements to assure inserting of possible alarm source coding and start-stop coding and reason coding for projected occurrences. It also specified the sensor walk around checkout procedure by a man carrying a shovel to activate the magnetic systems as well as the recording of synoptic and environmental conditions. Tests were conducted in planned segments, the longest of which was 24 hours. At least two periods of thundershowers were recorded, with durations of 15 minutes and 4 hours respectively.

A special close range helicopter test was planned to determine quantitatively the influence of helicopters on the sensors at large. Since false alarm coincidence had been associated with aircraft and helicopters flying in the vicinity it was felt that the quantitative data would be significant in clarifying the false alarm results. Straight flight passes were planned from minimum altitudes, 100 feet, to 2,000 feet, over and beyond the field at both minimum and high speeds. Sensor coverage was obtained by flying three paths, over the center and over each edge in both north/south and east/west patterns. The sensor and recording instrumentation were to be operated as in false alarm testing with run number, path, speed, altitudes, parameters as basic inputs. Helicopter position on entering and leaving the range boundry was to be called out on each pass for relating alarm response to position.

Data Reduction and Analysis

The flow of sensor data from the tests is shown in Figure 4. Upon receipt of test run data from the range the test logs were reviewed with the test engineer and any pertinent comments added to the log. The logs were filed in a summary log book, the chart records were catalogued by test date and the digital tape records transmitted to the computer building with a copy of the test log.

The test data reduction was performed primarily by the computer. Hand reduction of the parallel recorded chart data was performed regularly on a sample basis to verify the basic alarm counts for both false alarm and detection.

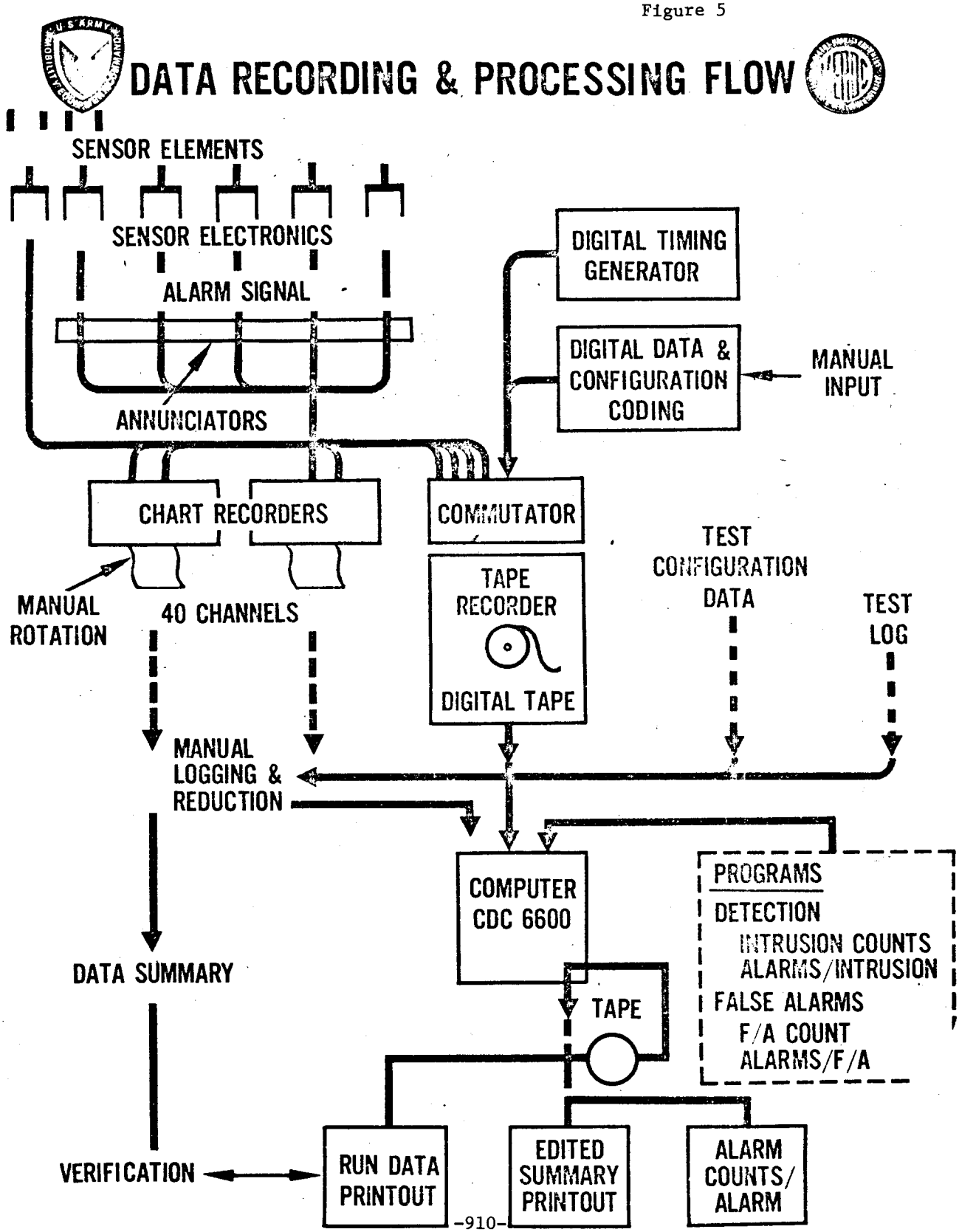
The computer detection program whose configuration is shown in Figure 5 data format provided a printout of individual stake pass data for each sensor including the run configuration and conditions the pass and detection counts for the stake and the computed results. The computations performed were: the probability of detection for the number of passes at the stake, the total alarm count, maximum, minimum and average, and the cumulative probability distribution of alarms (alarms per alarm), the minimum, maximum and average pass time and the minimum, maximum and average detection time duration. At the completion of the stake by stake data-printout, a summary for the test was formulated which totalled the individual stake data and provided the performance of the sensor.

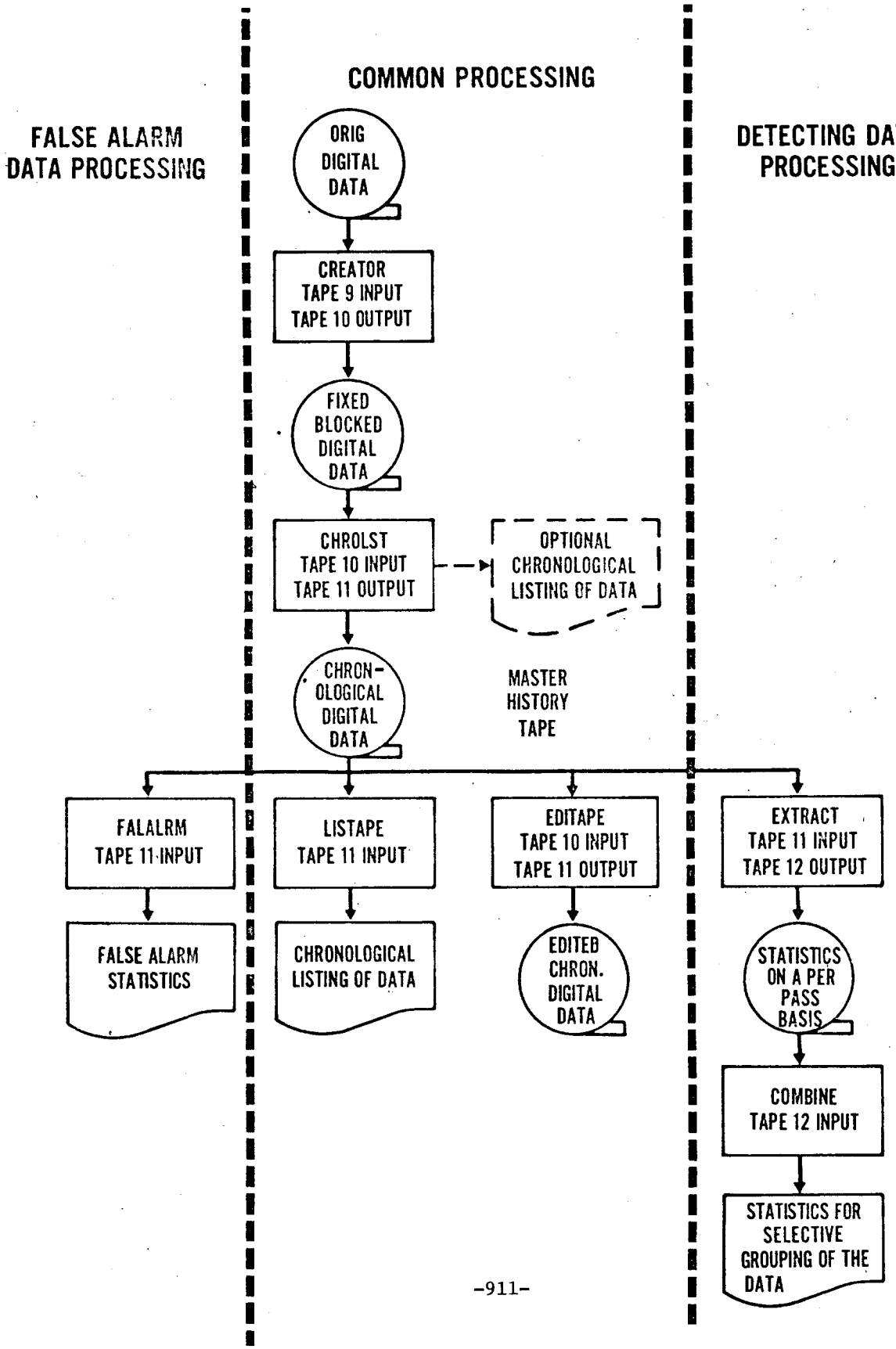
A second summary program provided for collecting the individual accumulated stake data from the several tests performed at different times.

The false alarm printout included: a chronological listing of the sensor alarms and the recognized potential alarm sources as inserted by code; a matrix of alarms by sensor in 15 minute intervals for four hour test periods with accompanying false alarm subtotals by sensor; and summary matrix of alarms by sensor in one hour intervals, with accumulated totals for each sensor.

In assembling the data for expressing the cumulative probability of detection for a sensor, the intent was to provide an average probability of detection. This could reflect the sensor performance assuming equal possibilities for any of the intruder modes being employed. Normally, the number of passes for each intruder configuration and speed were equal, so that this intent was realized. This was particularly important for the magnetically clean man are largely different, and a large number of passes in one condition, or the other, can shift the average probability number significantly.

Figure 5


DATA RECORDING & PROCESSING FLOW
SENSOR ELEMENTS
SENSOR ELECTRONICS
ALARM SIGNAL
ANNUNCIATORS
DIGITAL TIMING GENERATOR
DIGITAL DATA & CONFIGURATION CODING
MANUAL INPUT
CHART RECORDERS
COMMUTATOR
MANUAL ROTATION
40 CHANNELS
TAPE RECORDER
DIGITAL TAPE
TEST CONFIGURATION DATA
TEST LOG
MANUAL LOGGING & REDUCTION
COMPUTER CDC 6600
PROGRAMS
DETECTION
INTRUSION COUNTS
ALARMS/INTRUSION
FALSE ALARMS
F/A COUNT
ALARMS/F/A
DATA SUMMARY
TAPE
VERIFICATION
RUN DATA PRINTOUT
EDITED SUMMARY PRINTOUT
ALARM COUNTS/ ALARM


COMMON PROCESSING
FALSE ALARM DATA PROCESSING
DETECTING DATA PROCESSING
ORIG DIGITAL DATA
CREATOR TAPE 9 INPUT TAPE 10 OUTPUT
FIXED BLOCKED DIGITAL DATA
CHROLST TAPE 10 INPUT TAPE 11 OUTPUT
OPTIONAL CHRONOLOGICAL LISTING OF DATA
CHRON-OLOGICAL DIGITAL DATA
MASTER HISTORY TAPE
FALALRM TAPE 11 INPUT
LISTAPE TAPE 11 INPUT
EDITAPE TAPE 10 INPUT TAPE 11 OUTPUT
EXTRACT TAPE 11 INPUT TAPE 12 OUTPUT
FALSE ALARM STATISTICS
CHRONOLOGICAL LISTING OF DATA
EDITED CHRON. DIGITAL DATA
STATISTICS ON A PER PASS BASIS
COMBINE TAPE 12 INPUT
STATISTICS FOR SELECTIVE GROUPING OF THE DATA

TEST RESULTS

Typical test results for magnetic and balanced pressure line sensors are shown, to indicate the degree of reduction and analysis performed. Figure 6 and 7, are representative of the magnetic sensor test result. Figure 6 shows the matrix of 5 intruder modes and configurations evaluated. Figure 7 demonstrates the comprehensive method of presenting this data.

In contrast the reduced number of modes and configurations required for evaluating BPS sensors is indicated in Figure 8. This is reflected in the more simple Pd display in Figure 9.

EXAMPLE I - MAGNETIC SENSOR TEST CONFIGURATION

SPEED OF INTRUDER	MAGNETIC CONFIGURATION
SLOW WALK (2 FT/SEC)	CLEAN
NORMAL WALK (4 FT/SEC)	NORMAL
RUNNING (8 FT/SEC)	CLEAN WITH WEAPON OVERHEAD
SPRINTING (10 FT/SEC)	
CRAWLING (¼-1 FT/SEC)	

1 MAN MAKING INTRUSION PERPENDICULAR TO SENSOR

Figure 6

EXAMPLE I

MAGNETIC SENSOR TEST RESULTS

MAGNETIC CONFIGURATION	SPEED OF INTRUDER	NO. OF PASSES	P_D
CLEAN	CRAWLING	28	.1
	SLOW		0
	NORMAL		0
	RUNNING		0
	SPRINTING		0
NORMAL	CRAWLING	28	1.0
	SLOW		1.0
	NORMAL		1.0
	RUNNING		1.0
	SPRINTING		1.0
WEAPON OVERHEAD	CRAWLING		.34
	SLOW		.04
	NORMAL		0.0
	RUNNING		.46
	SPRINTING		.833

Figure 7

EXAMPLE II

BALANCED PRESSURE SYSTEM TEST CONFIGURATION

SPEED OF INTRUDER	WEIGHT OF INTRUDER
SLOW WALK	142 LB
NORMAL WALK	210 LB
RUNNING	
SPRINTING	
CRAWLING	

Figure 8

EXAMPLE II
BALANCED PRESSURE SYSTEM TEST RESULTS

WEIGHT OF INTRUDER	SPEED	NO. PASSES	P_D
142 LBS	CRAWLING	28	.7
	SLOW WALK		.8
	NORMAL WALK		.7
	RUNNING		.6
	SPRINTING		.5
210	CRAWLING	28	.8
	SLOW WALK		.9
	NORMAL WALK		.8
	RUNNING		.7
	SPRINTING		.6

Figure 9

LABORATORY CONTROL OF DYNAMIC VEHICLE TESTING

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Suspension & Track Functions
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ABSTRACT. In order to study vehicle suspension and frame dynamics under controlled and reproducible laboratory conditions, TACOM's road simulator or "shaker test" was developed. A road simulator is a laboratory test device which imparts dynamic forces simulating road inputs, on a complete vehicle. It is the purpose of this study to develop vertical position control signals for the road simulator so that good correlation between laboratory test and field results is obtained.

As a result of this study, the design engineer has a more exact vehicle model than he has had and the test engineer has a laboratory simulation which has been verified for vertical dynamic inputs. The combined effect of these two engineering tools will serve to produce a better prototype vehicle which, in turn, will eliminate many of the initial field test failures which plague new vehicles.

INTRODUCTION. The road simulator concept of laboratory vehicle testing came into existence to facilitate studies of frame and suspension dynamics. Prior to the road simulator, frame and suspension components were divorced from the vehicle for laboratory evaluation. In most cases, the control or excitation signal for the test was some well defined mathematical function whose correspondence validity to actual field excitation is questionable. Testing then progressed to a point where recorded field signals and shaped random noise were used to control component tests.

Since there is interaction between the component being tested and the vehicle to which it is mounted, it became apparent that a road simulator which would test the total vehicle system in the laboratory would yield useful results. The early road simulators provided vertical inputs of low amplitude to each wheel of a passenger car. The inputs were accomplished using four electro-hydraulic linear actuators with pedestals on which the tires rested.

Laboratory testing of off-road vehicles offered a new challenge. Due to the large wheel deflections, the vehicle had to be restrained from falling off the road simulator. In order to facilitate restraining the vehicle and also to allow the addition of longitudinal

excitation forces, the wheels were removed and the spindles attached to the actuator through a multiple degree of freedom assembly. The restraints were attached from vehicle to ground so that their effect on the dynamic motion of the sprung mass was minimal.

The present state of the art includes vehicles in the 5-ton payload class with up to six vertical and four horizontal linear actuators. Currently being constructed at the U.S. Army Tank-Automotive Command (TACOM) is a road simulator for 1/4-ton class vehicles which has four vertical actuators with position control, four horizontal actuators with load control and four rotary hydraulic pumps to be used as absorption dynamometers with torque control. This simulator, fully operational, will test the total vehicle system under controlled laboratory conditions.

The analog position of force signals which control the electro-hydraulic actuators must produce motions or forces in the vehicle which can be correlated with those which were recorded during field tests. This paper will present in detail three different techniques by which valid position control signals may be obtained from recorded field data or from surveyed terrain elevations.

CONTROL SIGNAL GENERATION TECHNIQUES. The analog signal which controls the electro-hydraulic actuators of the road simulator may be proportional to either position or force. The vertical road input actuators are position controlled and the fore and aft horizontal road input actuators are load controlled. The most readily obtained field data which can be transformed into vertical wheel spindle displacements are vertical accelerations of the wheel spindles. Terrain profile data can also be transformed into vertical wheel spindle displacement if accurate vehicle and tire models are on hand.

The field recorded fore and aft loads require no transformation if the actuators are load controlled. Force is proportional to acceleration, so the following analyses are applicable.

Double Integration

The acceleration signal which is to be double integrated is the vertical acceleration of the wheel spindle. The vertical accelerations of each wheel are recorded simultaneously so that the control signals generated from these accelerations will have the proper phase relationship. The acceleration signals thus recorded are a function of the suspension geometry, the suspension parameters, the tire characteristics and the terrain profile. Changes in any of the above vehicle characteristics would require that a new test course traverse be made. For the following analysis, assume that an accurate recorded acceleration signal is available.

The integral of well defined mathematical functions can be found in any calculus text. Mathematically, the integral of a continuous random variable such as vertical wheel acceleration is also well defined. In fact, the double integral of acceleration which results in displacement is also well defined. The physical implementation of double integration, however, is not well defined.

It is a well known fact that a stable perfect double integrator which has the transfer function $G_1(S) = 1/S^2$ is not physically realizable. The task, then, is to develop a stable transfer function, the frequency response of which approaches a double integrator in the desired frequency band of .5 to 50 Hz. This band is within the response limits of most road simulators and also includes the frequencies of interest for suspension dynamics studies. The transfer function chosen to double integrate the recorded acceleration signal to produce the position control signal for the road simulator is $G_2(S) = \frac{KS^2}{(S+4)^4}$

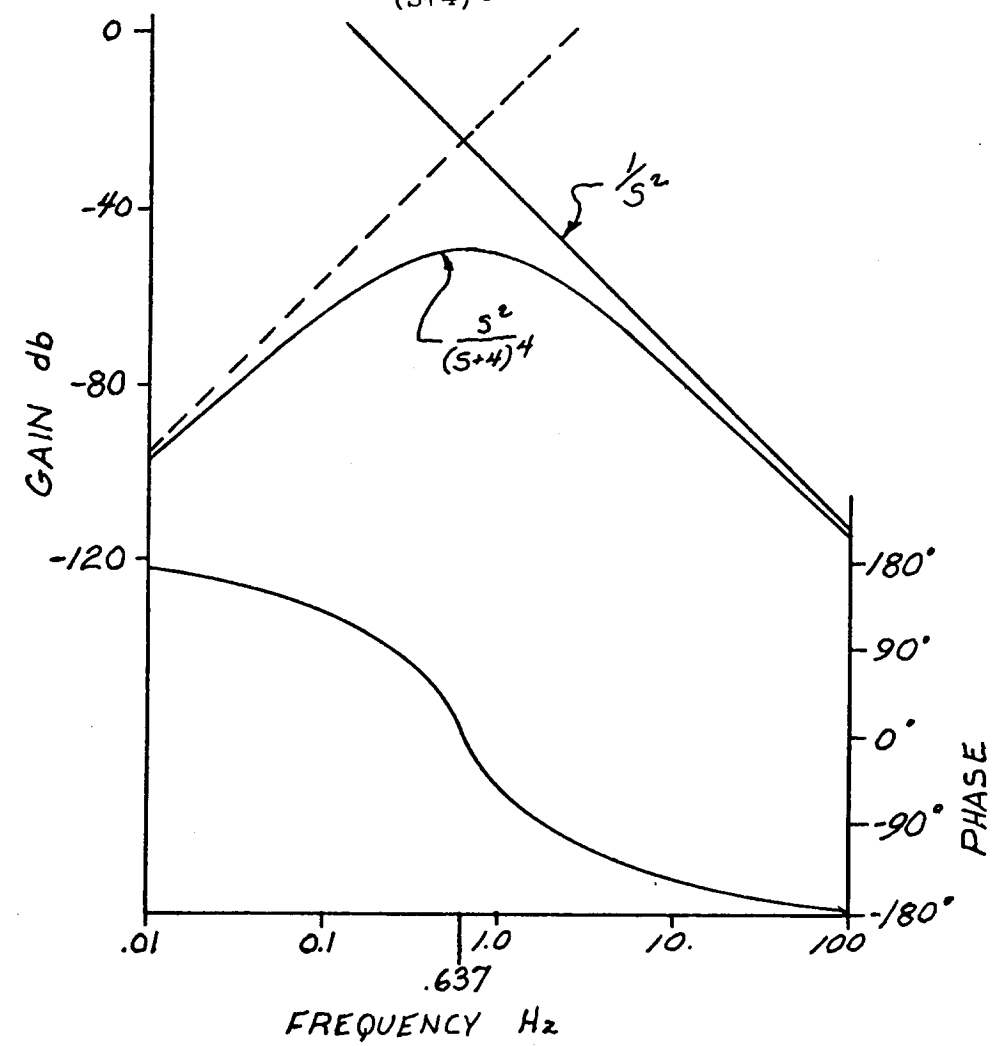

FIGURE 1. BODE PLOT

As can be seen in Figure 1, the frequency response curve for $G_2(S)$ when K = 1 asymptotically approaches perfect double integration beyond .636 Hz. The important characteristic of $G_2(S)$ is that the low frequency components of the input signal are suppressed. These low frequency components, especially zero frequency of D.C. offset, are the prime contributors to unstable double integration. This fact is quite clear in Figure 1. As frequency approaches zero, $G_1(S)$ approaches infinity and $G_2(S)$ approaches zero. The low frequency accuracy of $G_2(S)$ can be theoretically improved by shifting the intersection of its asymptotes to the left. This can be accomplished by decreasing the constant 4 in the denominator to 3, for example. However, as this constant approaches zero, $G_2(S)$ approaches $G_1(S)$. Another way to increase low frequency accuracy is to increase K. Increasing K, however, raises the whole response curve and the higher frequency accuracy decreases. Stability is also reduced as K is increased. Either method requires trial and error to determine which K or which denominator gives acceptable response in the desired frequency range.

Figure 2 shows the result of playing a field-recorded acceleration signal into $G_2(S)$ with K = 1. The acceleration signal was recorded at the front wheel spindle of an M656 5-ton 8x8 cargo truck as it traversed the Aberdeen Proving Ground Belgian Block Course at an average speed of 15 miles per hour. The displacement signal peak to peak magnitude of 12 feet (Figure 2) was observed during the recording of the acceleration tape.

The accuracy of the approximate double integration depends, of course, upon the transfer function used to perform this operation. The correlation between the field recorded vertical acceleration of the wheel and the laboratory recorded vertical acceleration of the wheel can be computed to numerically determine the accuracy of the double integration.

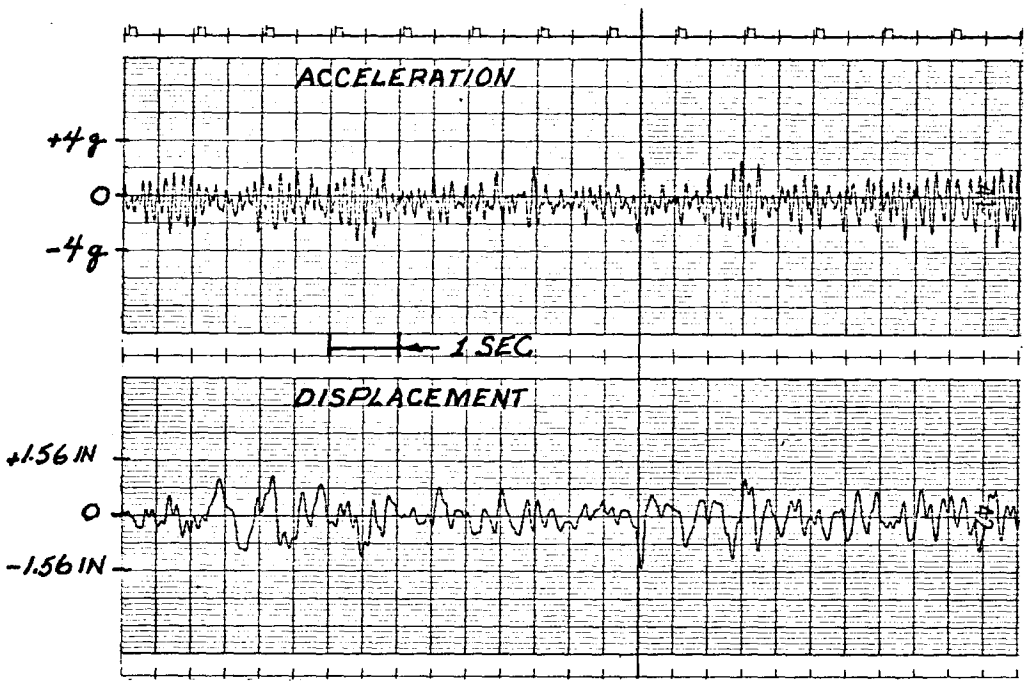

FIGURE 2. ACCELERATION & DISPLACEMENT-TIME CURVES

Filtered Noise

The idea of playing random noise, which has a flat power spectrum, through a shaping filter to control a laboratory road simulator has been suggested previously (references 1 and 2).

In order to apply this control technique, the vertical wheel spindle acceleration must be recorded on magnetic tape during field runs. From this data, a shaping filter for random noise is desired such that the filter output is a displacement signal statistically equivalent to the recorded acceleration signal.

Let the filter to be defined be a linear time invariant function so that conventional methods of analysis may be used. The total system is:

$$y(t) = h(t) \times n(t) \tag{1}$$

Where--

n(t) is the random noise input
h(t) is the filter
y(t) is the output displacement

Using the convolution integral and fourier transform, as described in reference 3, page 182, the following relationship is obtained from equation (1):

$$S_{dd}(f) = \left|H(j2\pi f)\right|^2 \cdot S_{nn}(f) \tag{2}$$

Where--

$S_{dd}(f)$ is the power spectral density (PSD) of the desired displacement control signal

$S_{nn}(f)$ is the PSD of random noise. This is a constant and will be defined to be unity

$H(j2\pi f)$ is the frequency response function for the shaping filter

The relationship between displacement and acceleration PSD's is defined to be:

$$S_{dd}(f) = \frac{1}{(2\pi f)^4} \cdot S_{aa}(f) \tag{3}$$

Substituting $S_{dd}(f)$ in equation (3) gives:

$$S_{aa}(f) = (2\pi f)^4 \left|H(j2\pi f)\right|^2 S_{nn}(f) \tag{4}$$

Since $S_{nn}(f) = 1$ by previous definition:

$$H(j2\pi f) = \frac{1}{(2\pi f)^2} \cdot \sqrt{S_{aa}(f)} \tag{5}$$

Equation (5) will now be used to obtain from field data the frequency response for the desired filter.

Figure 3 is a PSD curve of the vertical front wheel acceleration of an M656 5-ton 8x8 cargo truck. The acceleration signal was recorded during field tests as Aberdeen Proving Ground at an average vehicle speed of 14.2 miles per hour. The test courses were the Belgian Block Course, Three-Inch Spaced Bump Course, Two-to-Four-Inch Radial Washboard Course, Imbedded Rock Course and Two-Inch Washboard Course. Substituting the values for $S_{aa}(f)$ from Figure 3 into Equation (5) results in the desired frequency response curve shown in Figure 4. The desired curve was approximated using an ESIAC algebraic computer by the following transfer function where $j2\pi f$ is replaced by the Laplace Operator S:

$$H(S) = \frac{15.123\ (S^2 + 86.4\ S + 202S)}{(S^2 + 10.5S + 122S)\ (S^2 + 5S + 25)} \qquad (6)$$

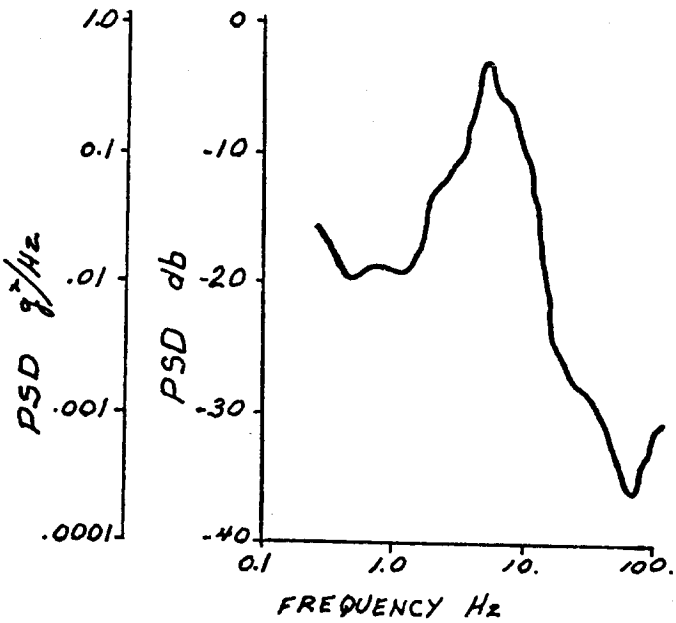

FIGURE 3. PSD OF FIELD RECORDED VERTICAL WHEEL ACCELERATION

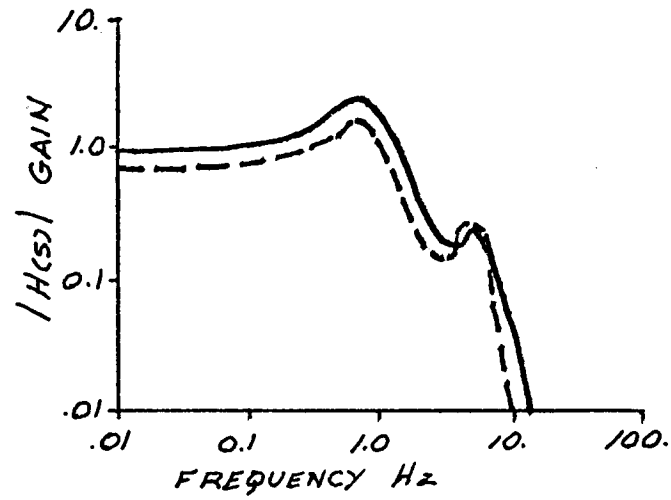

FIGURE 4. FREQUENCY RESPONSE CURVES FOR DESIRED AND ACTUAL FILTERS

The actual frequency response curve for Equation (6) is the dashed curve in Figure 4. Figure 5 is the output of the filter with a random noise input.

Since PSD is an approximate measurement and the actual filter response function is an approximation of the desired filter response function, the accuracy of this technique depends upon the accuracy of the approximations. This technique is validated using statistical measurement techniques such as histograms, cross correlation and probability density functions.

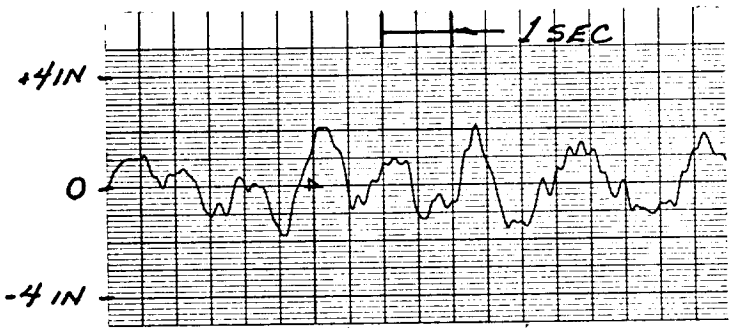

FIGURE 5. POSITION CONTROL SIGNAL

Terrain Profile

The two previous methods of control signal generation required that vehicle dependent acceleration signal at the wheel spindle be recorded during test course traverse. In other words, new instrumented test runs must be made for each different vehicle configuration. Consider now the possibility of using surveyed terrain profiles to control the road simulator system where the excitation is through the wheel spindle as previously stated.

Surveyed terrain elevation data are readily available, in reference 4 for example. The major problem to be solved then is the transfer function from terrain to the wheel spindle. This transfer function represents not only the tire assembly dynamics but is also a function of the suspension dynamics and the sprung mass. It is concluded then that a mathematical model of the total vehicle system is required. This model would be programmed on an analog or hybrid computer and run in parallel with the road simulator to provide the wheel spindle position control signal. The block diagram of this system is shown in Figure 6.

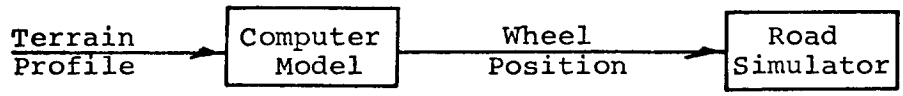

FIGURE 6

System Block Diagram

The system in Figure 6 assumes an accurate model of both the tire and the vehicle. The tire is a complex non-linear system which is discussed thoroughly in reference 5. A tire model can be made so complex that it is unwieldy or it can be simplified to a second order mass-spring-damper system. The latter case with a realistic non-linear spring and point follower, Figure 7, may give satisfactory results for the vertical control of a road simulator.

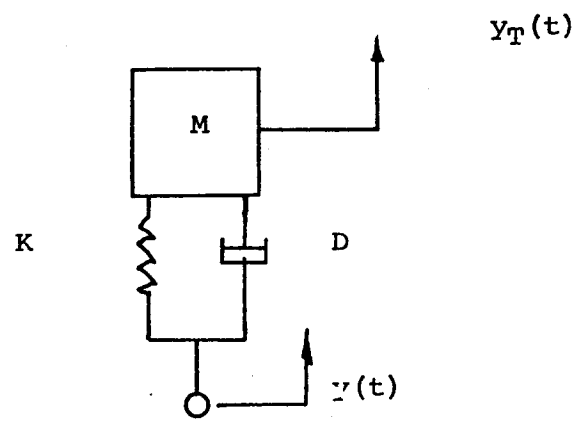

FIGURE 7

Simple Tire Model

Where--

M is the unsprung mass
K is the spring rate
D is the damping coefficient
y(t) is the terrain profile
$y_T(t)$ is the wheel displacement

NOTE that the velocity profile, $\dot{y}(t)$, of the terrain is also required. The digitized terrain profile is digitally differentiated to obtain $\dot{y}(t)$.

Obtaining an accurate mathematical model of the vehicle dynamics is facilitated by the availability of the road simulator. The differential equations of motion for the vehicle are obtained using any of the conventional techniques such as Lagrangian or Newtonian mechanics. The equations are then programmed on an analog or hybrid computer. The computer model and the road simulator are excited at the wheel spindles with identical signals and the responses are compared. The response is a combination of sprung mass output signals, which could include, for example, pitch, bounce and roll displacements. The parameters of the computer model are adjusted either manually or automatically such that the error between comparison signals is minimized. Reference 6 presents a continuous parameter tracking technique which could be extended to attain the automatic parameter adjustment.

Once the accurate model is obtained, its parameters may be easily adjusted to maximize some index of performance such as driver comfort. The sensitivity of any performance parameter to changes in each of the physical vehicle parameters can be measured. This type of study tells the design engineer a range of acceptable values for each physical parameter. The physical parameters include spring rates, damping coefficients, center of gravity location, wheel base, etc.

The above described technique is an ambitious undertaking currently being implemented at TACOM. Extensive computer analysis is required, but the resulting vehicle model will give the design engineer a new tool with which to improve vehicle performance.

SUMMARY. Three methods for obtaining the road simulator control signal (the input at the wheel spindle of the test vehicle) have been presented.

The double integration of field recorded vertical wheel spindle acceleration and the filtered random noise both require field data acquired by a vehicle similar to the test vehicle. These two methods are well suited to long term durability studies.

The third method requires and facilitates the development of an accurate mathematical model of the vehicle. The selected terrain profile is played into the computerized vehicle model and the wheel displacement signals from the model are then used to control the road simulator.

The first two techniques have been used at TACOM to control simulators. The third technique is currently being implemented; we expect to be using it by December 1971.

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FUNCTIONAL PROPERTIES OF CSP-1 APPLIED TO A FINITE LENGTH PRODUCTION RUN

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1. INTRODUCTION

In 1943 Harold Dodge [5.1] presented the first continuous sampling plan, which has come to be known as CSP-1. One of the underlying assumptions in the mathematical formulation of this plan is that it will be applied to an unending flow of units. Using this assumption, among others, such functional properties as Average Fraction Inspected (AFI), Average Outgoing Quality (AOQ), and Average Outgoing Quality Limit (AOQL) can be computed.

Cases occur in practice however, where the number of units for which an acceptance decision is to be made is neither unbounded nor even large enough that Dodge's formulae for the functional properties mentioned would apply. If continuous sampling is otherwise desirable in these cases, it is necessary to be able to properly determine plan parameters if sufficient, but not excessive, inspection is to be required.

Motivated then by both quality and cost considerations, formulae were developed to describe the functional properties of CSP-1 applied to a finite number of units. This paper provides the derivation of these formulae and some background discussion.

2.0 SUMMARY

When a CSP-1 plan is applied to a specified finite number N of units, AFI_N can be expressed as follows:

$$AFI_N = \frac{1}{N}\left(i + Jf + (1-f)\left((1-q^i + Jfp)\ \frac{1-Y^J}{1-Y}\right.\right.$$

$$\left.\left. - fp\ \frac{1 - (J+1)\ Y^J + J\ Y^{J+1}}{(1-Y)^2}\right)\right)$$

where

i = the clearance number,

f = the sampling frequency,

J = $N-i$,

p = the probability that a unit is defective,

q = $1-p$, and

Y = $[(1-fp)(1-q^i)-pq^i]/(1-q^i)$.

Under the assumption that defective units found are removed and replaced with good units, AOQ_N can be expressed as follows:

$$AOQ_N = p(1-AFI_N) .$$

The variance of outgoing quality for a plan applied to a finite number of units can be determined exactly by using the expression for AFI_N to derive solutions to the unknown portions of Lasater's [5.2] variance model. The complexity of the resulting formula for variance suggests that a digital computer would have to be employed to obtain numerical solutions.

3.0 DISCUSSION

3.1 Background.

3.1.1 Continuous sampling. When product in a plant is moving in a continuous flow, as for example on a conveyor, there are to be certain advantages to carrying out inspection also on a continuous basis as the product is flowing past the inspection station rather than forming lots and subsequently drawing a random sample from the lot. One of the most obvious advantages is that the discovery of a defective unit in the flow of product can provide a signal that a flaw may be present in the process. Necessary corrective action then can be initiated immediately. The sampling plan used to determine conformance of product thereby serves as a process control tool also.

Continuous sampling was the innovation of Harold Dodge, who in 1943 provided the mathematical rationale and the rules of operation for the first continuous sampling plan, which has come to be known as CSP-1. The initials stand for "Continuous Sampling Plan".

3.1.2 CSP-1. The rules of operation for CSP-1 are shown on Figure 1. At the start of inspection, the screening crew inspects 100% of the units. This 100% inspection is continued until some predetermined number, i, of consecutive units, are defect free.

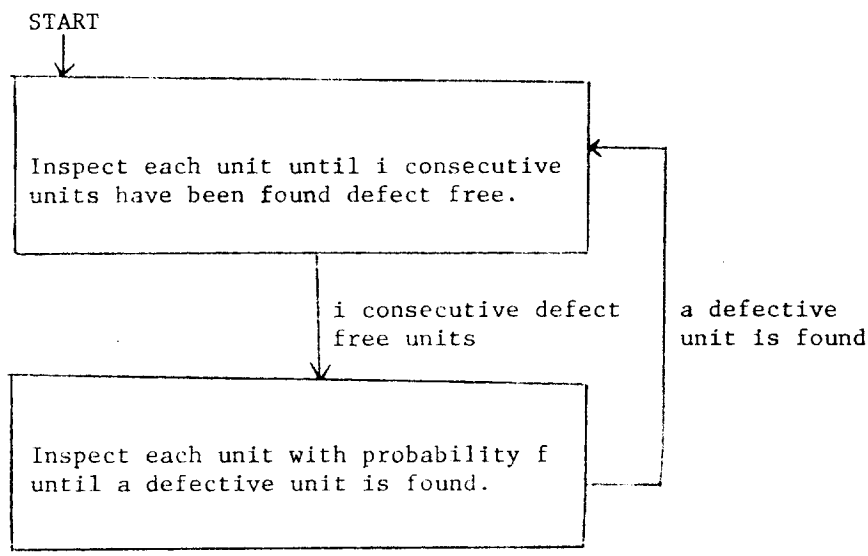
At this time, the screening crew is released from 100% inspection and the sampling inspector inspects a fraction, f, of the units, where the sample units are selected in a random manner. When the sampling inspector finds a defect we go back to the beginning of the cycle, and again the screening crew inspects 100% of the units. It can be seen then that CSP-1 consists simply of alternate sequences of screening, or 100% inspection, and sampling inspection.

For CSP-1, then, two parameters define the plan. The first of these is the clearance number, to be identified hereafter as i, the other is the sampling frequency, to be identified hereafter as f. For example, our sampling plan might have the parameter i equal to 200 and f equal to one-tenth.

3.1.3 Functional properties of a CSP-1 plan. Knowledge of certain properties of continuous sampling plans is necessary for intelligent use of the plans. The first of these properties is Average Fraction Inspected, or AFI, which can be defined as the expected value of the fraction of material that will be inspected over an indefinitely long period of time when each unit has

FIGURE 1

PROCEDURAL DIAGRAM FOR CSP-1



probability p of being defective.

The next property is Average Outgoing Quality, or AOQ, which can be defined as the expected fraction of material that is defective in accepted material over an indefinitely long period of time when each unit submitted for inspection has probability p of being defective.

The maximum value of the AOQ that can exist for a plan if we consider all values of p, is called the Average Outgoing Quality Limit or AOQL.

Notice that we have implied in the above definitions that p is constant over all units. Other properties can be described when p is considered a variable. However, we shall not discuss these other properties in this paper.

3.1.4 Dodge's expressions for functional properties. Dodge, under an assumption of an unending flow of units of product, and using mathematical methods based on geometric distributions and expected values, determined the expression for AFI for CSP-1 to be

$$AFI = \frac{f}{f(1-q^i) + q^i}$$

where $q = 1-p$.

The AOQ for each value of p can then be found simply as

$$AOQ = p(1-AFI)$$

when defective units found are removed and replaced with good units. This replacement assumption will be the only one considered in this paper, primarily so we will later be able to use the work of Lasater for checking purposes.

3.1.5 Evolution in methodology. Before proceeding, it will be appropriate here to digress for a moment to discuss the evolution in mathematical methodology associated with continuous sampling theory. We will mention only those works which have some connection with what will follow; we do not pretend that we are providing a thorough history of continuous sampling.

As mentioned, the original work in continuous sampling was carried out by Dodge. The next event of interest to us is the application of Markov chain theory to continuous sampling plan development, described in 1955 by Lieberman and Solomon [5.3]. The beauty and power of Markov chain methods led to many new developments; in fact, statisticians working in continuous sampling were inevitably drawn to learn the fundamentals of Markov Chain theory. A paper by S. W. Roberts [5.4] provided some Markov chain approaches to CSP-1 formula derivation. Lasater, using a Markov chain approach from Roberts, carried out some studies on CSP-1 applied over a finite number of units.

Our discussion will now run somewhat parallel to material presented by Lasater in his dissertation. The notation, to the extent possible, conforms to Lasater's although there are some differences.

Similarly, our Markov chain state construction resembles closely that of Lasater, although his chain was constructed to describe AOQ, whereas ours is constructed to describe AFI.

3.2 The Markov chain approach.

3.2.1 The states of the chain. Let us define a Markov chain with each state s of the chain corresponding to a unit of product that can be associated in a certain way with the operation of a CSP-1 plan.

Let

s_k = H(k-1)

for k=1, . . . ,i which denotes the case where 100% inspection is in effect and the unit corresponding to this state is the k^{th} unit in an attempt to clear i consecutive nondefective units.

s_{i+1} = SI

which denotes the case where sampling inspection is in effect and the unit corresponding to this state is selected for inspection.

s_{i+2} = SN

which denotes the case where sampling inspection is in

effect and the unit corresponding to this state is not selected for inspection.

It is seen that every possible situation (unit-wise) has been covered in a set of i+2 states.

3.2.2 The transitional probability matrix. Figure 2 provides the transitional probability matrix for the Markov chain corresponding to CSP-1 with states just described. An element of the matrix located in the j^{th} row and k^{th} column is the probability of going from the state identifying the j^{th} row to the state identifying the k^{th} column.

3.2.3 Some properties. Lasater has shown that the Markov chain he used to describe CSP-1 is discrete, finite, recurrent, irreducible and aperiodic. By definition, the Markov chain used in this paper has these same properties, and we will therefore make use of certain properties described by Lasater. First, let Π'^{k} denote the vector of state probabilities after k transitions. The elements of this row vector are then the probabilities of being in, respectively, HO, H1, . . . , H(i-1), SI and SN after k transitions, and

$$\Pi'^{n} = \Pi'^{n-1}\,\underline{P} = \Pi'^{n-2}\,\underline{P}^{2} = \ldots = \Pi'^{0}\,\underline{P}^{n}$$

where $\underline{P}^{j}$ is the j^{th} power of the transitional probability matrix $\underline{P}$. This property was used by Lasater, who provided numerical solutions to several CSP-1 examples by performing series of transformations on a digital computer.

FIGURE 2

TRANSITIONAL PROBABILITY MATRIX

	HO	H1	H2	.	.	.	H(i-2)	H(i-1)	SI	SN
HO	p	q	0	.	.	.	0	0	0	0
H1	p	0	q	.	.	.	0	0	0	0
H2	p	0	0	.	.	.	0	0	0	0
.	.	.	.				.	.	.	.
.	.	.	.				.	.	.	.
.	.	.	.				.	.	.	.
H(i-2)	p	0	0	.	.	.	0	q	0	0
H(i-1)	p	0	0	.	.	.	0	0	fq	(1-f)q
SI	p	0	0	.	.	.	0	0	fq	(1-f)q
SN	0	0	0	.	.	.	0	0	f	(1-f)

Lasater's numerical solutions therefore provided a good checking device for our derivations, as will be explained later.

Further, let $\lim_{n\to\infty} \Pi'^{n} = \Pi'$, that is, let Π' be defined as the row vector of limiting probabilities. Then $\Pi'\underline{P} = \Pi'$ for the type of Markov chain described here. That is, the row vector of limiting probabilities is transformed into itself by the transitional probability matrix.

The derivation that we shall now provide for the AFI of a CSP-1 plan applied only to some specified finite number of units will make use of the algebraic equality expressed in this last property, disregarding, however, what gave rise to the equality. In other words, given the vector of limiting probabilities (regardless of how we got it!), we know that the transitional probability matrix transformation will have no effect on the elements of this vector, because of the equality described above.

3.3 Derivation of AFI_N and AOQ_N

3.3.1 The elements of the row vector of limiting probabilities. In order to proceed on to our derivation, we must first solve for the elements of the row vector of limiting probabilities. Using the usual algorithms for solving this vector, such as those given by Roberts, for example, one easily finds that

$$\Pi' = (\Pi(H0), \Pi(H1), \ldots, \Pi(H(i-1)), \Pi(SI) \quad \Pi(SN))$$

$$= \left(\frac{p}{D}, \frac{pq}{D}, \ldots, \frac{pq^{i-1}}{D} \quad \frac{fq^i}{D} \quad \frac{(1-f)q^i}{D} \right)$$

where $D = f(1-q^i) + q^i$.

We next have need to determine the ratio of the probability of occupying a state associated with 100% inspection to the probability of occupying a state associated with sampling, either inspected or skipped.

Because of disjointness, the probability of the union of H's can be found by addition to be

$$\text{Prob} \left(\bigcup_{j=0}^{i-1} Hj \right) = \frac{f(1-q^i)}{f(1-q^i) + q^i}$$

In a similar fashion,

$$\text{Prob}\ (SI \cup SN) = \frac{q^i}{f(1-q^i) + q^i} \quad .$$

Finally, the desired ratio is found to be

$$\left(\text{Prob} \left(\bigcup_{j=0}^{i-1} Hj \right)\right) / \text{Prob}\ (SI \cup SN) = f(1-q^i)/q^i \quad .$$

3.3.2 Definition of AFI_N. For a CSP-1 plan applied to a finite number n of units, define AFI_N as the Average Fraction Inspected of those N.

For $N \leq i$, AFI_N is 1.

For $N > i$, let us define $J=N-i$, so $N=i+J$. By definition, AFI_N is $\frac{1}{N}$ (Expected Number Inspected) or $\frac{1}{N}$ (i + Expected Number Inspected of last J).

3.3.3 The derivation. Let us now consider the operation of a CSP-1 plan. The first i units will be free of defects with probability q^i, in which case sampling can be initiated. The first i units will contain at least one defective unit with probability $1-q^i$, in which case screening must be continued. What we want to attempt to do now is segment these probabilities and treat at least part of the problem here. We will attempt to do any segmenting in such a fashion that for at least part of the problem we can work with the vector of limiting probabilities.

We can proceed as shown in Table I, by segmenting $1-q^i$ into two portions, $f(1-q^i)$ and $(1-f)(1-q^i)$. Following segmentation, all of the states associated with screening have the same relationship with each other as they do in the vector of limiting probabilities. The segmenting by the factor f would act on each of these probabilities in the same manner. The result is that now relative to both the state probabilities associated with sampling

TABLE I

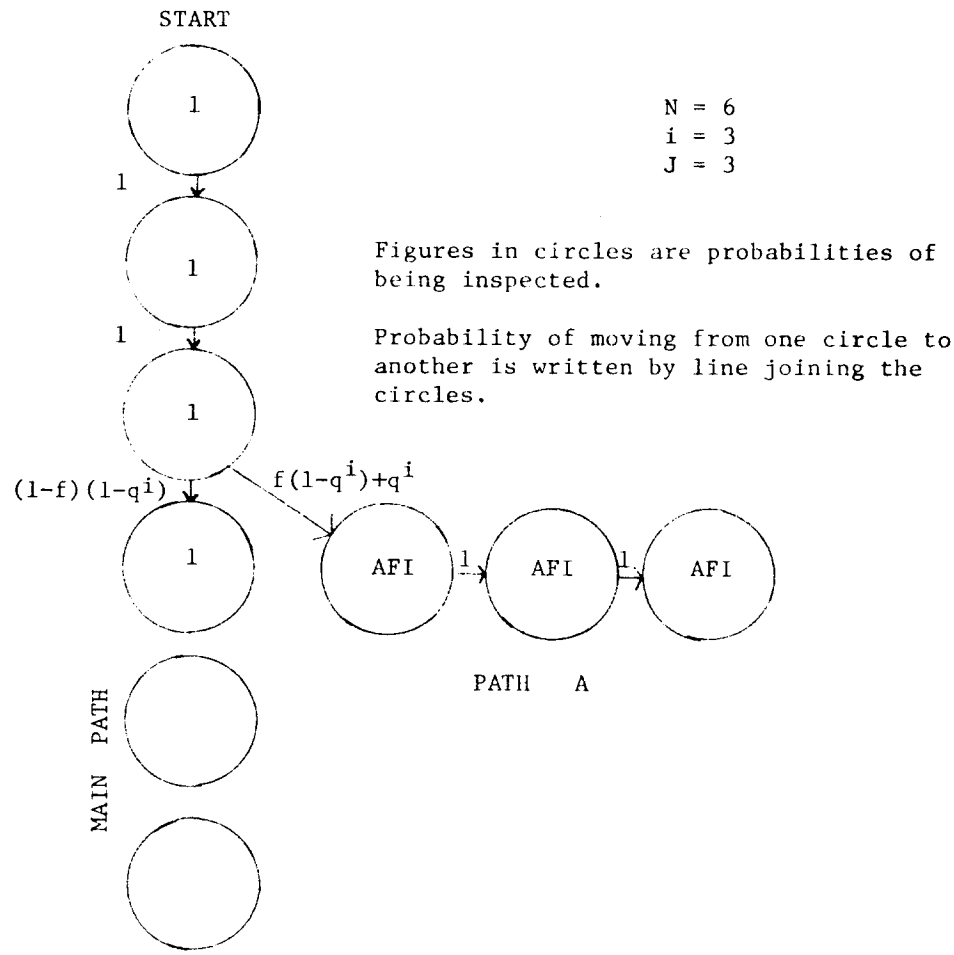
SEGMENTATION AT (i+1)st UNIT

Screening Continues	$f(1-q^i)$	+	$(1-f)\ (1-q^i)$
Sampling Initiated	q^i		
	$f(1-q^i)+q^i$		$(1-f)\ (1-q^i)$
	Enter Path A (See Fig. 3)		Stay on Main Path (See Fig. 3)

****

FIGURE 3

EXAMPLE OF PARTIAL SOLUTION OF PROBLEM



and that portion associated with screening segmented by the factor f, the relationship between state probabilities is the same as in the vector of limiting probabilities. Now, if we talk about probabilities given that we have entered Path A, say, (see Figure 3), we find that we are talking about the vector of limiting probabilities. The ratio $f(1-q^i)/q^i$ obtained previously has thus served as a guide for telling us how we must segemnt if we want to reach a point where part of the problem can be handled.

Now, since we have established that given Path A, the state probability vector will equal the limiting probability vector, and since the transitional probability matrix will transform this vector into itself, we see that the probability that a unit is inspected is constant over all units in the path and can easily be shown to be AFI (as defined for the infinite case).

For Path A only, then, the expected number of units becomes

$$J\ (f(1-q^i) + q^i)\ AFI$$

$$= J\ (f(1-q^i) + q^i)\ \frac{f}{f(1-q^i)+q^i} = Jf,$$

a result surprising for its simplicity.

Considering now also the portion treated in the main path, for J=1 we can now say

$$AFI_N = \frac{1}{N}\ (i+f+(1-f)(1-q^i))$$

which is what we obtain also if we treat the problem through more direct probabilistic considerations by considering all possible outcomes if we have one more unit to be inspected than the clearance number, i.

As the work thus far may have suggested, our strategy will be to go step-by-step down the main path, breaking off on side paths such as Path A as circumstances allow, until we are finished. However, one might well dread having to go through additional mathematics on a step-by-step basis, considering how long the problem might become. This is not necessary, because after the first i units have been inspected, the probability of initiating sampling is q^i. From the main path, from the (i+1)st unit on, the probability of initiating sampling from each point is $pq^i/(1-q^i)$, which is the probability of clearing i at that point given that screening was in effect for the i units preceding the unit in question.

Therefore, what we cover now is applicable from this point on. Since we will initiate sampling from the main path with probability $pq^i/(1-q^i)$, we will remain on screening with probability $1-pq^i/(1-q^i) = (1-q^i-pq^i)/(1-q^i)$, and this latter expression is the one we will want to segment in such a fashion that the relationship of the portion segmented to the quantity $pq^i/(1-q^i)$ is the same as the ratio $f(1-q^i)/q^i$. The portion to be segmented is then easily found to be $fp(1-q^i)/(1-q^i)$.

Table II shows how the segmenting is carried out. Figure 4 is a continuation of the illustrative example begun in Figure 3. To our previous answer, we can now add

$$(1-f)(1-q^i)\ \frac{p(f(1-q^i)+q^i)}{1-q^i}\ (J-1)AFI$$

$$= (1-f)(1-q^i)\ \frac{p(f(1-q^i)+q^i)}{1-q^i}\ (J-1)\ \frac{f}{f(1-q^i)+q^i}$$

$$= (1-f)(1-q^i)\ \frac{pf(J-1)}{(1-q^i)}\quad \text{for Path B.}$$

To handle going to the (i+2)nd unit on the main path, we can add

$$(1-f)(1-q^i)\ \frac{((1-fp)(1-q^i)-pq^i)}{(1-q^i)}$$

$$= (1-f)(1-q^i)Y, \text{ say, for ease of notation.}$$

Note that in Figure 4 we have also taken care of the (i+3)rd unit on the main path and Path C, since, as we have mentioned, the segmentation will occur as shown in Table II.

We are now able to generalize, using the concept illustrated in Figure 4 as a guide. We can describe the expression for AFI_N as

$$AFI_N = \frac{1}{N}\ (i+Jf+(1-f)(1-q^i)(\sum_{j=0}^{J-1} Y^J + \frac{fp}{1-q^i}\sum_{j=0}^{J-1}(J-j-1)Y^J))$$

TABLE II

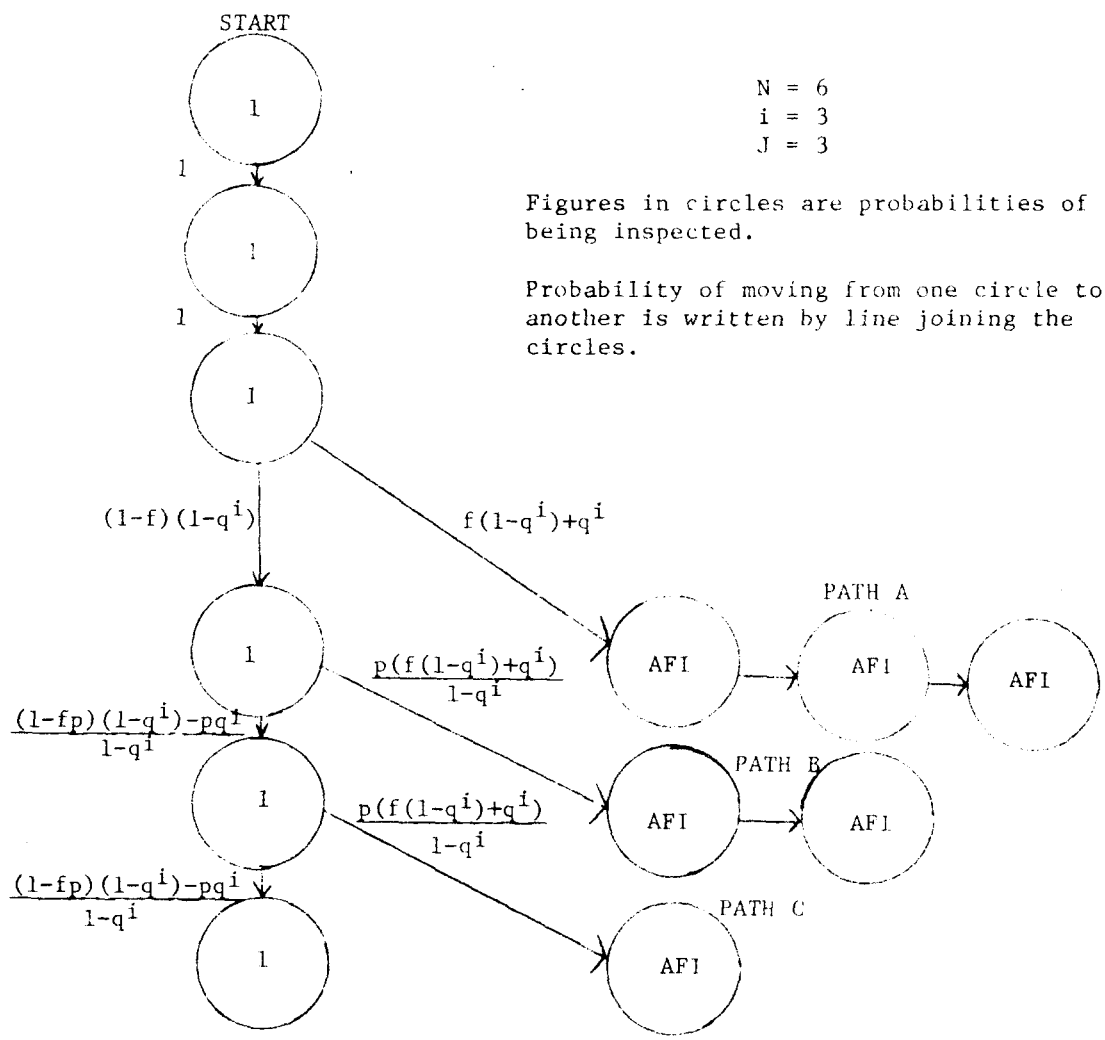
SEGMENTATION AT (i+2)nd UNIT

SCREENING CONTINUES	$fp(1-q^i)/(1-q^i)$	+ $((1-fp)(1-q^i)-pq^i)/(1-q^i)$
SAMPLING INITIATED	$pq^i/(1-q^i)$	
	$(fp(1-q^i)+pq^i)/(1-q^i)$ Enter Path B (See Fig. 4)	$((1-fp)(1-q^i)-pq^i)/(1-q^i)$ Stay on Main Path (See Fig. 4)

****

FIGURE 4

COMPLETE EXAMPLE



N = 6
i = 3
J = 3

Figures in circles are probabilities of being inspected.

Probability of moving from one circle to another is written by line joining the circles.

Simplifying, by using geometric progression relationships, we obtain

$$\sum_{j=0}^{J-1} Y^j = \frac{1-Y^J}{1-Y}$$

$$\sum_{j=0}^{J-1} (J-j-1)Y^j = J \sum_{j=0}^{J-1} Y^j - \sum_{j=0}^{J-1} (j+1)Y^j$$

$$J \sum_{j=0}^{J-1} Y^j = J \frac{1-Y^J}{1-Y}$$

$$\sum_{j=0}^{J-1} (j+1)Y^j = \sum_{j=0}^{J-1} \frac{d}{dY} Y^{j+1} = \frac{d}{dY} \frac{Y-Y^{J+1}}{1-Y}$$

$$= \frac{1 - (J+1)Y^J + JY^{J+1}}{(1-Y)^2}$$

So AFI_N

$$= \frac{1}{N} (i+Jf+(1-f)(1-q^i)((1+ \frac{Jfp}{1-q^i}) \frac{1-Y^J}{1-Y}$$

$$- \frac{fp}{1-q^i} \frac{1-(J+1)Y^J+JY^{J+1}}{(1-Y)^2}))$$

$$= \frac{1}{N} (i+Jf+(1-f)((1-q^i+Jfp) \frac{1-Y^J}{1-Y}$$

$$- fp \frac{1-(J+1)Y^J+JY^{J+1}}{(1-Y)^2})) \quad .$$

We can describe AOQ_N as simply $AOQ_N = p(1-AFI_N)$.

3.3.4 Checks. Because of the many steps taken to arrive at the result shown above, a method of verification seemed prudent. Three different kinds of checks were carried out.

First, for J from zero through two, the problem was attacked using direct probabilistic methods by considering possible results along with their probabilities of occurrence, etc. and constructing an expression for AFI_N. This check was successful. The rapid increase in complexity of this checking method as J increases limits its value. Of course, this check proves nothing for higher values of J.

The next check was to find the limit of our derived expression for AFI_N as N (and thus J also) approaches infinity; the limit was found to be Dodge's expression for AFI, as it should be.

Finally, the numerical values of AOQ_N provided by Lasater in his dissertation via curves were compared with $AOQ_N=p(1-AFI_N)$ for several different plans and several values of p, and found to be equal.

3.4 Variance of outgoing quality.

3.4.1 Background. Under the assumption of an unending flow of units, no concern need be expressed over the variance of outgoing quality, since the variance will be zero when considered over an infinite number of units. However, if a plan is to be applied over a specified finite number of units, variability of outgoing quality may be of concern; it is conceivable that

outgoing quality might be "considerably" poorer than the computed AOQ_N, or it might be "considerably" better. Lasater derived an expression for variance of outgoing quality; however, as he pointed out, two unknowns still remained in his expression. With the expression for AFI_N just obtained, we are able to complete his expression. The result is somewhat complicated; digital computer solutions will be required.

3.4.2 The model. Lasater's model for the variance of outgoing quality can be expressed as follows:

$$\mathrm{Var}(OQ_N) = \Big(\sum_{m=1}^{N} p\, P^m_{HO,SN} \,(1-p\, P^m_{HO,SN}) + 2p^2 \sum_{m<r=2} \sum P^m_{HO,SN} \,(p^{r-m}_{SN,SN} - P^r_{HO,SN})\Big)/N^2$$

where $P^Z_{X,Y}$ is defined as follows: if at some time we are in state X, we will be in state Y after Z transitions with probability $P^Z_{X,Y}$.

The unknown quantities are $P^k_{HO,SN}$ and $P^k_{SN,SN}$. We will now derive expressions for these quantities.

3.4.3 $P^k_{HO,SN}$. We are concerned with the probability that if we are in state HO we will be in state SN after k transitions.

Without loss of generality, we can consider the short production run situation described heretofore.

Then $P^k_{HO,SN}$ can be considered to be the probability that the (k+1)st unit is skipped, which is

$$= 1 - \text{Prob ((k+1)st unit is inspected)}$$
$$= 1 - [(k+1)AFl_{k+1} - k\,AFl_k] .$$

3.4.4 $P^k_{SN,SN}$. We will now derive an expression for the probability that if we are in state SN we will be in state SN after k transitions. Unfortunately this is a more complicated problem than was the case for $P^k_{HO,SN}$.

It is given that the starting point is a skipped unit. After the first transition, we know that we are in one of the states associated with sampling: inspecting, or in state SI, with probability f, and skipping, or in state SN, with probability (1-f). Consider the following mutually exclusive events: a defect is first found on unit j, j=1, . . ., k, or no defects are found at all. (Notice that here we are talking about k rather than k+1 units since it is given that the first unit is skipped.)

Let us now define a new function, denoted $AFI_j(S)$, which shall be the average fraction inspected over j units given that the first unit falls in a sampling sequence and is inspected or skipped with appropriate probabilities.

Then clearly, using the same kind of reasoning as was used previously,

$$p^{k}_{SN,SN} = 1 - [k\ AFI_k(S) - (k-1)AFI_{k-1}(S)].$$

It will therefore only be necessary to derive an expression for $AFI_j(S)$.

Let B_h be the expected number of units inspected through the h^{th}, given that a defect was found on the h^{th} and no defects were found on the first h-1. This is found to be

$$\sum_{g=0}^{h-1} \binom{h-1}{g} \left(\frac{fq}{1-fp}\right)^{g} \left(\frac{1-f}{1-fp}\right)^{h-1-g} + 1$$

$$= (h-1)\frac{fq}{1-fp} + 1 \quad .$$

Now, if we should find a defect on the h^{th} unit in a sequence of k units, we may consider that the (h+1)st unit will correspond to state H0. Therefore, the average fraction inspected among the last k-h units is simply AFI_{k-h}.

We can then formulate $AFI_j(S)$ as follows:

$$AFI_j(S) = \frac{1}{j}\left[\left(\sum_{h=1}^{j} (B_h + (j-h)AFI_{j-h})(1-fp)^{h-1}fp\right)+(B_{j+1}-1)(1-fp)^j\right] .$$

It can be seen that the last term on the right hand side merely takes care of the case where no defects are found at all in the j+1 units.

Having thus obtained expressions for the $P^Z_{X,Y}$, the variance of OQ_N can be obtained.

4.0 FURTHER REMARKS

The results contained in this paper will permit sets of parameters to be developed for continuous sampling plans for a specified number of units. These sets of parameters can be constructed to yield a desired set of AOQL's; this has not been possible with previously developed finite length plans, such as CSP-A contained in MIL-STD-1235(ORD).

Perhaps some extension of the segmentation technique described in the paper will permit description of the finite properties of continuous sampling plans more complicated in structure than CSP-1.

It is also likely that there exists Markov chain application areas other than sampling plans wherein the segmentation technique could be used to describe finite properties of a process.

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THE USE OF A DESIGN OF EXPERIMENT FOR DETERMINING OPTIMUM BRIGHTNESS OF PHOSPHORS EXPOSED TO HIGH-ENERGY ELECTRON-BEAM BOMBARDMENT

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ABSTRACT

A design of experiment was prepared to plan and analyze experiments for a research and development program investigating the high-energy bombardment of phosphors. A 3^3 factorial design was used to identify the optimum conditions for obtaining the maximum brightness from specific phosphors that were exposed to a high-energy electron beam. The factors varied in the experiments were beam voltage, beam current density, and phosphor thickness. Quadratic expressions derived from the experimental measurements were obtained from a computer program using a Yates procedure of data analysis. Contour surfaces in a three-dimensional factor space indicating the optimum conditions for a number of particular phosphor compositions are presented.

The remainder of this paper was reproduced photographically from the manuscript submitted by the author.

THE USE OF A DESIGN OF EXPERIMENT FOR DETERMINING OPTIMUM BRIGHTNESS OF PHOSPHORS EXPOSED TO HIGH-ENERGY ELECTRON-BEAM BOMBARDMENT

INTRODUCTION

The purpose of this investigation of high-energy electron bombardment of phosphors was to identify the optimum conditions for obtaining the maximum brightness from phosphors. This information will be used to develop high-brightness cathode-ray tubes (CRT's) for viewing at ambient conditions of high brightness for projection cathode-ray tubes, and for a potential optical pumping source for laser excitation.

Electron-beam excitation of phosphors with the generation of light (i.e., cathodoluminescence) is a complex process that isn't fully understood. There exist experimental studies, however, that provide plausible explanations of what happens when an electron beam strikes a phosphor.

An experimental design was used to generate a functional relationship for brightness and thus obtain a better understanding of catholuminescence. Exploratory experiments were made to screen out unimportant factors so that following experiments would involve fewer variables. This was done with three-level Graeco-Latin square designs where electron-beam voltage, beam current density, and phosphor screen thickness were selected as the final factors. A 3^3 factorial design was used and a computer program was written to obtain quadratic expressions for brightness with the Yates procedure of data analysis. The empirically derived quadratic expression and the actual data were compared. Computer derived contour surfaces of brightness were derived for two different phosphors - P2(ZnS:Cu) and P24 (ZnO).

BACKGROUND

There are several theories in the literature that explain the complex processes in catholuminescence when a beam of high-energy electrons bombards a phosphor. Garlick[1] identifies the first process as the creation of back-scattered electrons because of primary-electron reflections and low-energy secondary electrons. Since the substrate of the phosphor samples in this study were 400 volts positive with respect to the anode of the electron gun, all back-scattered electrons were collected in the phosphor material. Lehmann[2] describes the process of producing excitation transition, which consists of penetrating electrons creating a cascade production of secondary electrons and localized excited states in phosphor crystals. Popov[3] states that freed electrons in the conduction band and holes in the valence band initially distribute themselves in a "nonequilibrium manner." This changes to an equilibrium distribution near the bottom of the conduction band and near the top of the valence band by thermal equalization transferring part of the initial excitation energy to the crystal lattice. The time for thermal equalization is about 10^{-12} second. The charge carriers will diffuse through the crystal; nonradiative centers, trapping centers, and luminescence centers compete for the carriers[4]. Light output will be produced when electrons recombine with holes that were previously captured by the luminescence centers, as described by Dekker[5].

In spite of the above work, no theoretical model has yet been developed for brightness as functions of excitation conditions and phosphor thickness, since most of the materials studied have been microcrystalline phosphors that make the interpretation of data difficult because of the complex geometry[5]. In 1903, Lenard[6] found that brightness and voltage can be related by the following expression:

$$L \propto (V-V_o)^q$$

where V_o is the dead voltage with q assumed to be unity (it may be greater than two).

Einstein[7] states that there exists a linear relationship between luminous output and current density for current density levels up to 10^{-2} A/cm^2 for ZnO and ZnS-CdS phosphors with a beam voltage of 50 kV. Tarlick[8] shows that luminescence intensity per unit area of a phosphor screen is proportional to current density for small values of current density (e.g., 200 $\mu A/cm^2$ for ZnS-Ag phosphor).

As one can see from the above, the theory of cathodoluminescence hasn't yet produced the mathematics that would describe the optimum conditions for light output.

APPARATUS

A demountable vacuum system is shown in Fig. 1 with an electron gun to the right and a high-capacity vacuum pump providing a vacuum of 10^{-8} Torr to the left. The system is all metal except for the light-output observation window, the isolation phosphor sample holder, and the electron-gun glass envelope extension (see Fig. 2). The phosphor sample is kept at 400 volts above the surrounding grounded anode in order to collect secondary electrons. The entire vacuum system is encased in 1/8 inch of lead for X-ray protection of nearby personnel. A mirror system transmits the phosphor light-output for the displaced aperture to prevent direct X-ray irradiation.

The angle of incidence for the electron beam with respect to the plane of the phosphor sample surface is 45 degrees. The excited or luminous phosphor surface is viewed at 45 degrees with respect to the plane of the phosphor sample surface. The dimensions of the projected area of the excited phosphor surface are measured with a cathetometer (a short-range telescope supported on a micrometer movement). The average brightness in foot-lamberts (fL) of the projected area of the excited phosphor surface is measured with a Spectra Brightness Spotmeter, which is calibrated with a 100 foot-lambert "standard" brightness source. The average brightness is computed from the readings of the spotmeter by applying a correction for the optical transmission attenuation factor (i.e., 70%) and the area ratio of the reticule in the brightness meter to the projected excited area (i.e., 31.8).

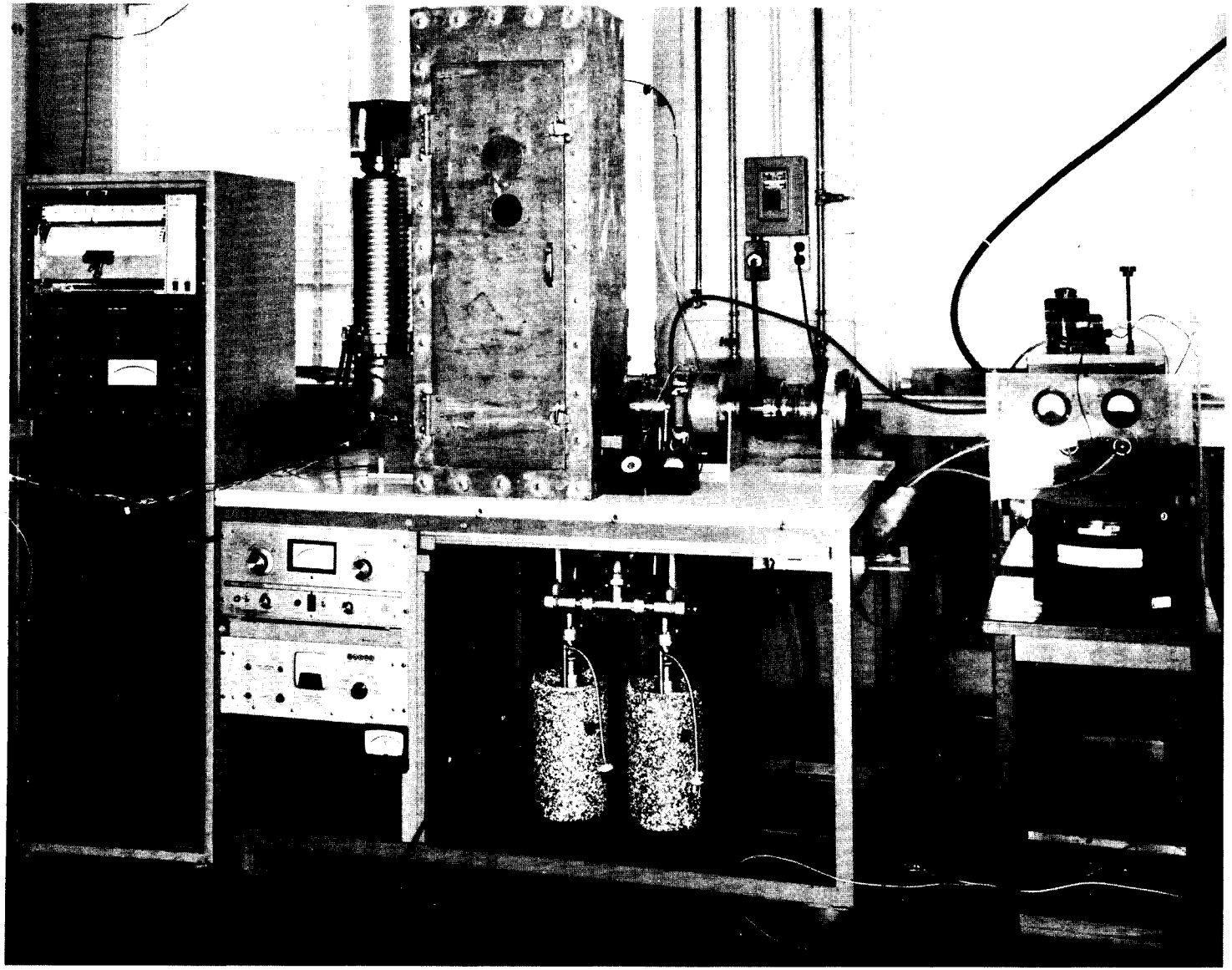

Fig 1 DEMOUNTABLE VACUUM SYSTEM

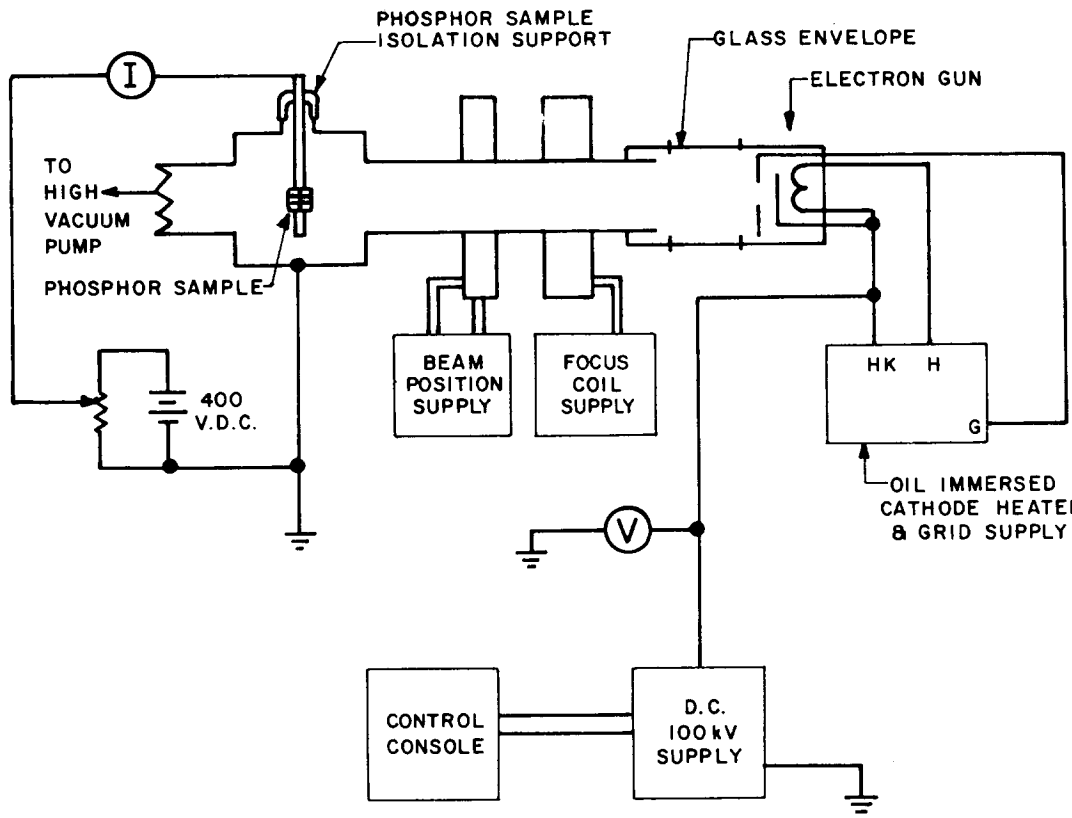

FIG 2 SCHEMATIC FOR PHOSPHOR BOMBARDMENT

TABLE I GRAECO-LATIN SQUARE DESIGN WITH AREA DENSITY OF 6.8 mg/cm^2

WATER - FLOW RATE (cc / MIN)	VOLTAGE (kV)		
	10	20	30
0	5/5 (11)	10/10 (38)	15/1 (55)
500	15/10 (12)	5/1 (18)	10/5 (31)
1000	10/1 (18)	15/5 (38)	5/10 (29)

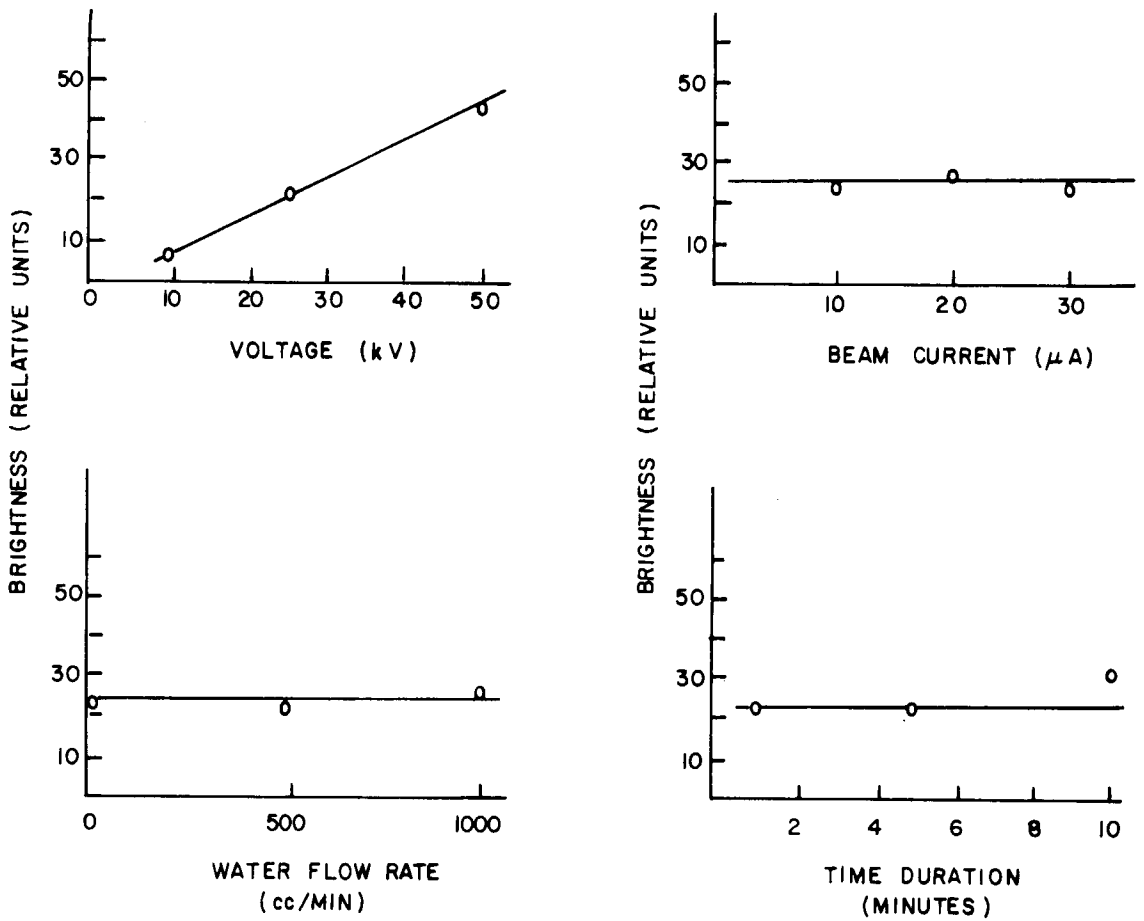

FIG 4 RESULTS FROM GRAECO-LATIN SQUARE DESIGN WITH AREA DENSITY OF 11.0 mg/cm^2

The above preliminary data analysis indicates that water-flow rate and time duration don't need to be considered in the following experiments.

There exists evidence in the literature of phenomena (such as, ionization, self-absorption of light, and thermal effects) that wouldn't make the remaining factors independent of each other. That is, there is a possibility of joint effects or interaction between factors. One isn't able to estimate interaction effects if one uses traditional methods of experimentation by variating one of the factors and keeping others constant. In order to define optimum or maximum light-output response, three-levels would be needed to obtain a quadratic function for brightness. Therefore, a 3^3 factorial design for each phosphor composition was selected with voltage, beam current density, and area density as factors.

P2 and P24 phosphors were selected because they represent two different types of phosphor - a sulfide phosphor and an oxide phosphor. Each copper substrate contains six areas that are available for beam bombardment. All six areas have a phosphor layer assumed to be of uniform thickness. The units of phosphor thickness or area density are expressed in milligrams per square centimeter. Each area is randomly selected for beam bombardment and it is used twice, since the assumption is made that each brightness measurement is independent of the previous measurement in the same location. The 12 measurements of each copper sample consist of 9 measurements needed for factorial design and the 3 replicate measurements are used to provide error estimates caused by any possible nonuniformity of phosphor thickness or deleterious effects of previous measurements. Three sets of 12 measurements supply the experimental design for each phosphor with 27 data values of the required 3^3 factorial design and with replicated tests of measurement error and measures of phosphor uniformity for each copper sample.

The data with replication for P2 phosphor, are given in Table III. The factors are represented by capital letters and the three levels by corresponding small letters with the numbers -1, 0, 1 as suffixes. The factors have the same notation and units as in the previous Graeco-Latin square design except that brightness is now expressed in foot-lamberts and area density is represented by T. The levels for voltage are 20, 40 and 60 kV; the levels for beam current density are 100, 175, and 250 $\mu A/cm^2$; and the levels for phosphor and density are 2.4, 7.7, and 12.6 mg/cm^2.

The above information was processed by a computer using the Yates procedure for a 3^3 factorial design. The results are given in Table V and Table VI. The coefficient column provides the coefficients in the quadratic function for brightness. Since there is no complete direct estimate of the experimental error variance, the combination of the three-factor interactions was used to form an estimate of error, since the higher order effects are assumed to be negligible.

TABLE III LIGHT YIELD IN FOOT-LAMBERTS FOR P2 PHOSPHOR

		LEVELS OF FACTOR T								
		t_{-1}			t_0			t_1		
		LEVELS OF FACTOR V			LEVELS OF FACTOR V			LEVELS OF FACTOR V		
		V_{-1}	V_0	V_1	V_{-1}	V_0	V_1	V_{-1}	V_0	V_1
LEVELS OF FACTOR J	j_{-1}	16218	35298	33398 30528	40068	55332	76320 77274	38160	95400	85860 85860
	j_0	28620	42930 45792	43884	47700	90630 89676	85960	54378	104940 100170	104940
	j_1	39114 41022	47700	49608	49608 50562	85860	100170	76320 74412	95400	114480

TABLE IV LIGHT YIELD IN FOOT-LAMBERTS FOR P24 PHOSPHOR

LEVELS OF FACTOR J		LEVELS OF FACTOR T								
		t_{-1}			t_0			t_1		
		LEVELS OF FACTOR V			LEVELS OF FACTOR V			LEVELS OF FACTOR V		
		V_{-1}	V_0	V_1	V_{-1}	V_0	V_1	V_{-1}	V_0	V_1
	j_{-1}	858	9540	8586	1431	8586	8586	763	9540	5724
				8586			7632			6678
	j_0	1431	11448	8586	1908	9063	6678	1431	9540	5247
			11448			7632			9540	
	j_1	2099	13356	8586	2671	7155	5724	1908	7632	4293
		1908			2862			1717		

TABLE V ANALYSIS OF THE 3 FACTORIAL EXPERIMENT FOR P2 PHOSPHOR

SOURCE OF VARIATION	COEFFICIENT	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SQUARE	VARIANCE RATIO
MAIN EFFECTS					
J LINEAR	10200	1.87×10^9	1		27.41^+
J QUADRATIC	-1320	9.35×10^7	1		1.38
V LINEAR	16800	5.10×10^9	1		74.78^+
V QUADRATIC	-4070	8.95×10^9	1		13.12^+
T LINEAR	23800	1.02×10^{10}	1		149.56^+
T QUADRATIC	-2960	4.73×10^8	1		6.93^*
TWO-FACTOR INTERACTIONS					
J_L V_L	-119	1.71×10^5	1		0.00
J_Q V_L	-252	2.28×10^6	1		0.03
J_L V_Q	1520	8.36×10^7	1		1.22
J_Q V_Q	1000	1.07×10^8	1		1.57
J_L T_L	994	1.19×10^7	1		0.17
J_Q T_L	-172	1.07×10^6	1		0.02
J_L T_Q	-225	1.83×10^6	1		0.03
J_Q T_Q	428	1.98×10^7	1		0.29
V_L T_L	8070	7.81×10^8	1		11.43^+
V_Q T_L	-1920	1.33×10^8	1		1.95
V_L T_Q	-2000	1.44×10^8	1		2.11
V_Q T_Q	-314	1.06×10^7	1		0.15
THREE-FACTOR INTERACTIONS		5.45×10^8	8	6.82×10^7	

+ HIGHLY SIGNIFICANT : $P \leq 1\%$
* SIGNIFICANT : $1\% < P \leq 5\%$

TABLE VI ANALYSIS OF THE 3^3 FACTORIAL EXPERIMENT FOR P24 PHOSPHOR

SOURCE OF VARIATION	COEFFICIENT	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SQUARE	VARIANCE RATIO
MAIN EFFECTS					
J LINEAR	-15.9	4.55×10^3	1		0.00
J QUADRATIC	-42.4	9.71×10^4	1		0.14
V LINEAR	2640	1.26×10^8	1		18340+
V QUADRATIC	-1740	1.63×10^8	1		237.20+
T LINEAR	-996	1.79×10^7	1		26.05+
T QUADRATIC	175	1.65×10^6	1		2.40
TWO-FACTOR INTERACTIONS					
J_L V_L	-652	5.10×10^6	1		7.42*
J_Q V_L	10.6	4.04×10^6	1		0.00
J_L V_Q	47.7	8.19×10^4	1		0.12
J_Q V_Q	59.4	3.74×10^5	1		0.54
J_L T_L	-644	4.98×10^6	1		7.23*
J_Q T_L	-60.9	1.34×10^5	1		0.02
J_L T_Q	199	1.42×10^6	1		2.10
J_Q T_Q	-18.5	3.72×10^4	1		0.01
V_L T_L	-811	7.89×10^6	1		11.48+
V_Q T_L	138	6.84×10^5	1		0.99
V_L T_Q	122	5.35×10^5	1		0.78
V_Q T_Q	-270	7.89×10^6	1		11.48+
THREE-FACTOR INTERACTIONS		5.50×10^6	8	6.8×10^5	

+ HIGHLY SIGNIFICANT : $P \leq 1\%$
* SIGNIFICANT : $1\% < P \leq 5\%$

The variance ratio or F-test shows that both phosphors have significant two-factor interactions indicating the factors aren't independent of each other. If one selects the coefficients whose variance ratios are significant and highly significant, one has the following functional relationship between brightness and the factors for P2 and P24 phosphors, respectively:

$$B = 64300 + 10200 \times (J-156)/39$$
$$+ 16800 \times (V-40)/20 - 4070 \times (V-40)^2/20^2$$
$$+ 23800 \times (T-6.2)/3.9 + 8070 \times (V-40) \times (T-6.2)/(3.9 \times 20)$$
$$- 2960 \times (T-6.2)^2/3.9^2$$

and

$$B = 5980 + 2640 \times (V-32.5)/25.5$$
$$- 652 \times (J-175) \times (V-32.5)/(27.5 \times 75)$$
$$- 1740 \times (V-32.5)^2/27.5^2 - 996 \times (T-7.7)/5.1$$
$$- 644 \times (J-175) \times (T-7.7)/(75 \times 5.1)$$
$$- 811 \times (V-32.5) \times (T-7.7)/(27.5 \times 5.1)$$
$$- 270 \times (V-32.5)^2 \times (T-7.7)^2/(27.5^2 \times 5.1^2)$$

where B is brightness,

J is beam current density

V is voltage, and

T is area density.

The empirical curve for brightness can be compared with the actual data in Fig. 5, 6, and 7 for P2 phosphor, and in Fig. 8, 9, and 10 for P24 phosphor. The points for the empirical curve were determined by a computer. Since the mathematical expression for P24 phosphor has more interaction terms than for P2 phosphor, one gets a better agreement between the empirical curve and the data points with P2 phosphor.

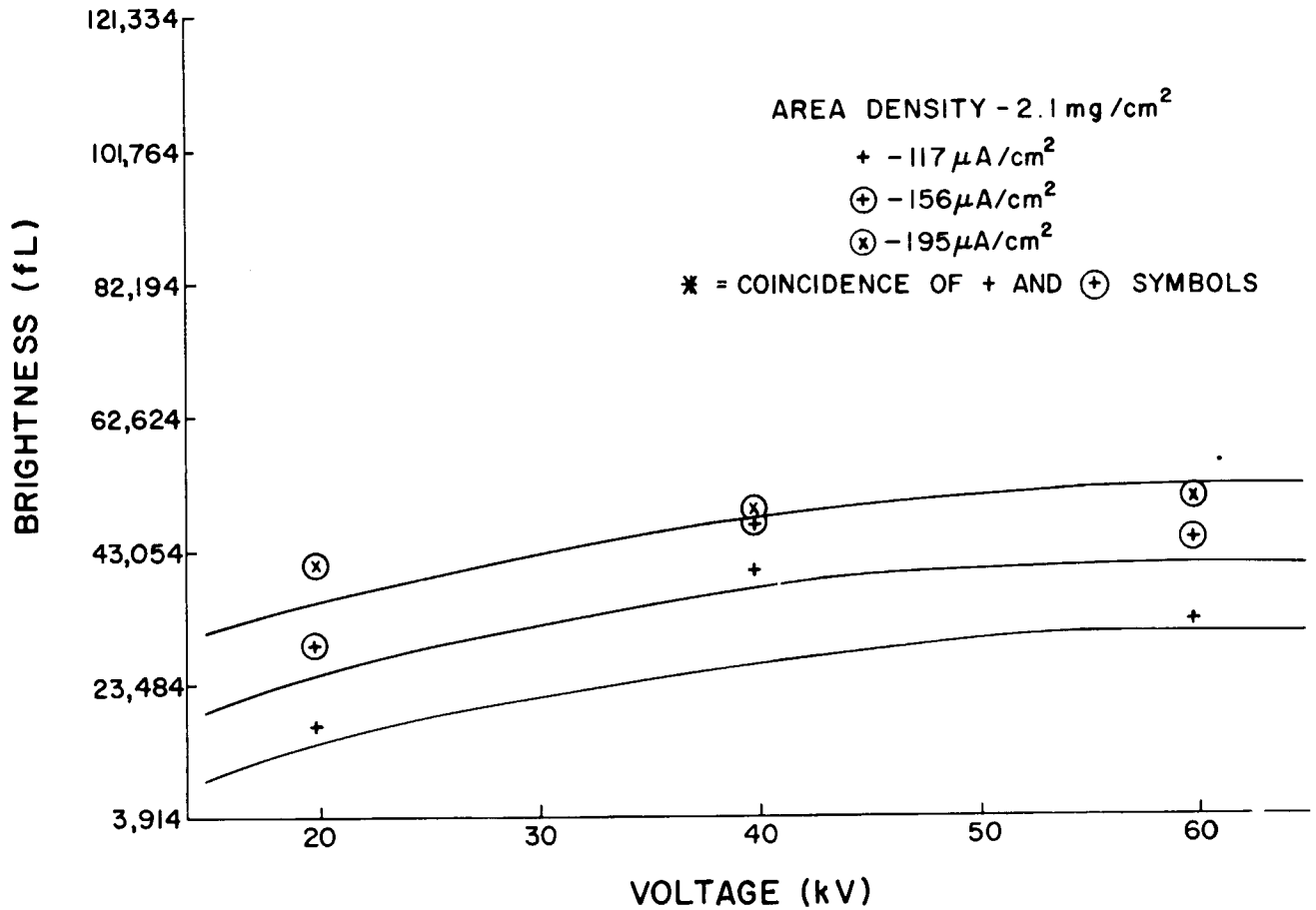

FIG 5 EMPIRICAL CURVES WITH EXPERIMENTAL DATA FOR P2 PHOSPHOR

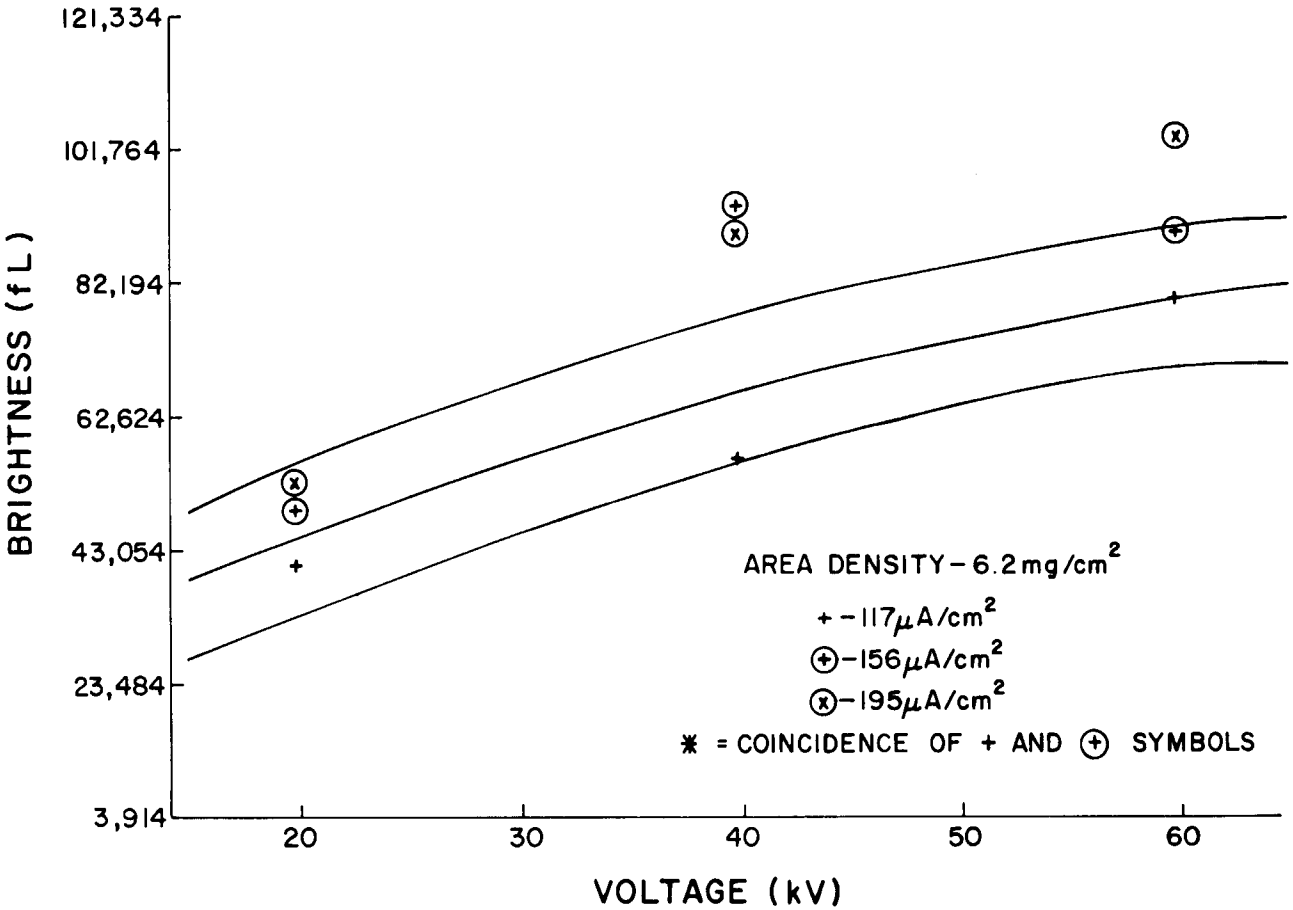

FIG 6 EMPIRICAL CURVES WITH EXPERIMENTAL DATA FOR P2 PHOSPHOR

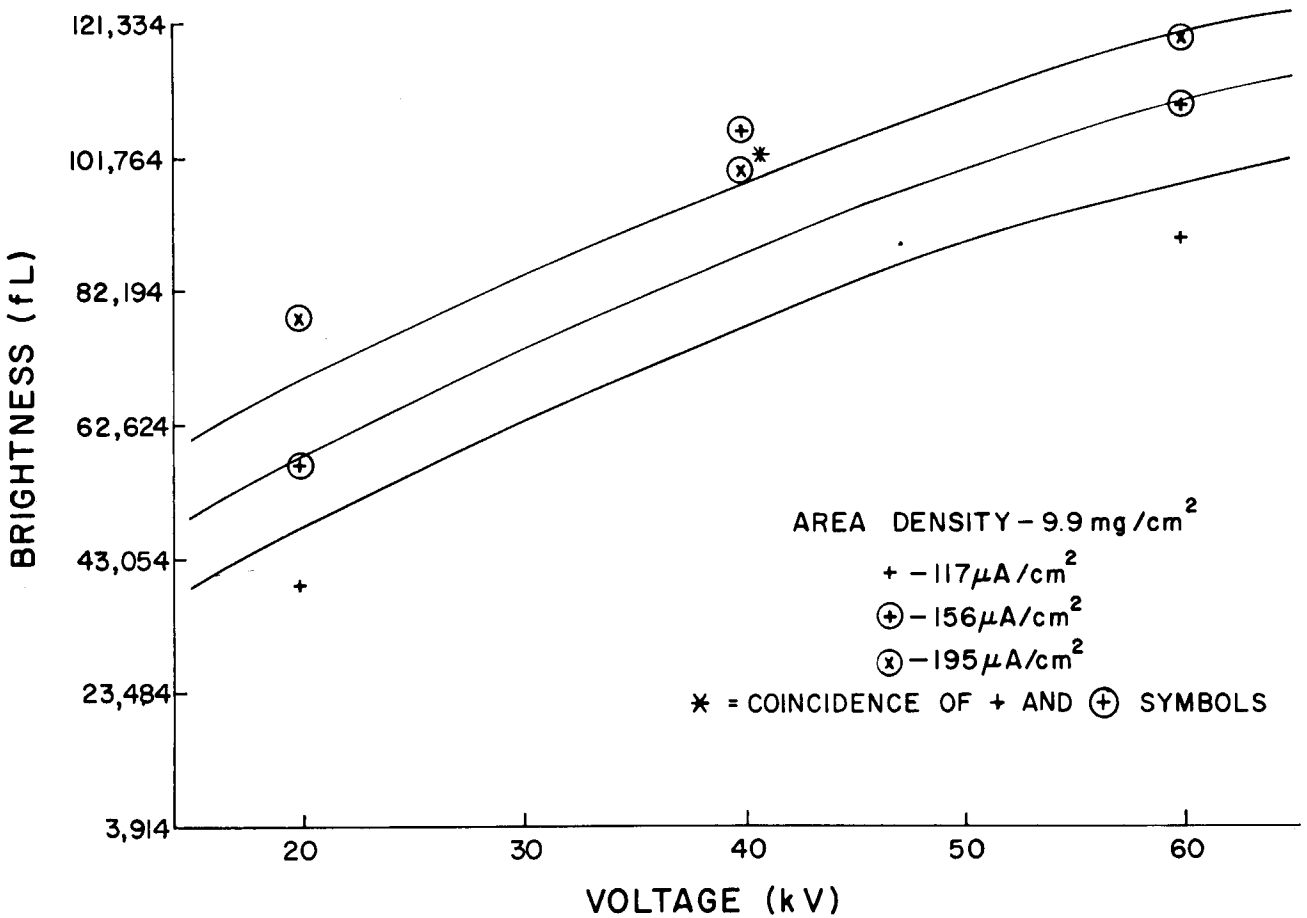

FIG 7 EMPIRICAL CURVES WITH EXPERIMENTAL DATA FOR P2 PHOSPHOR

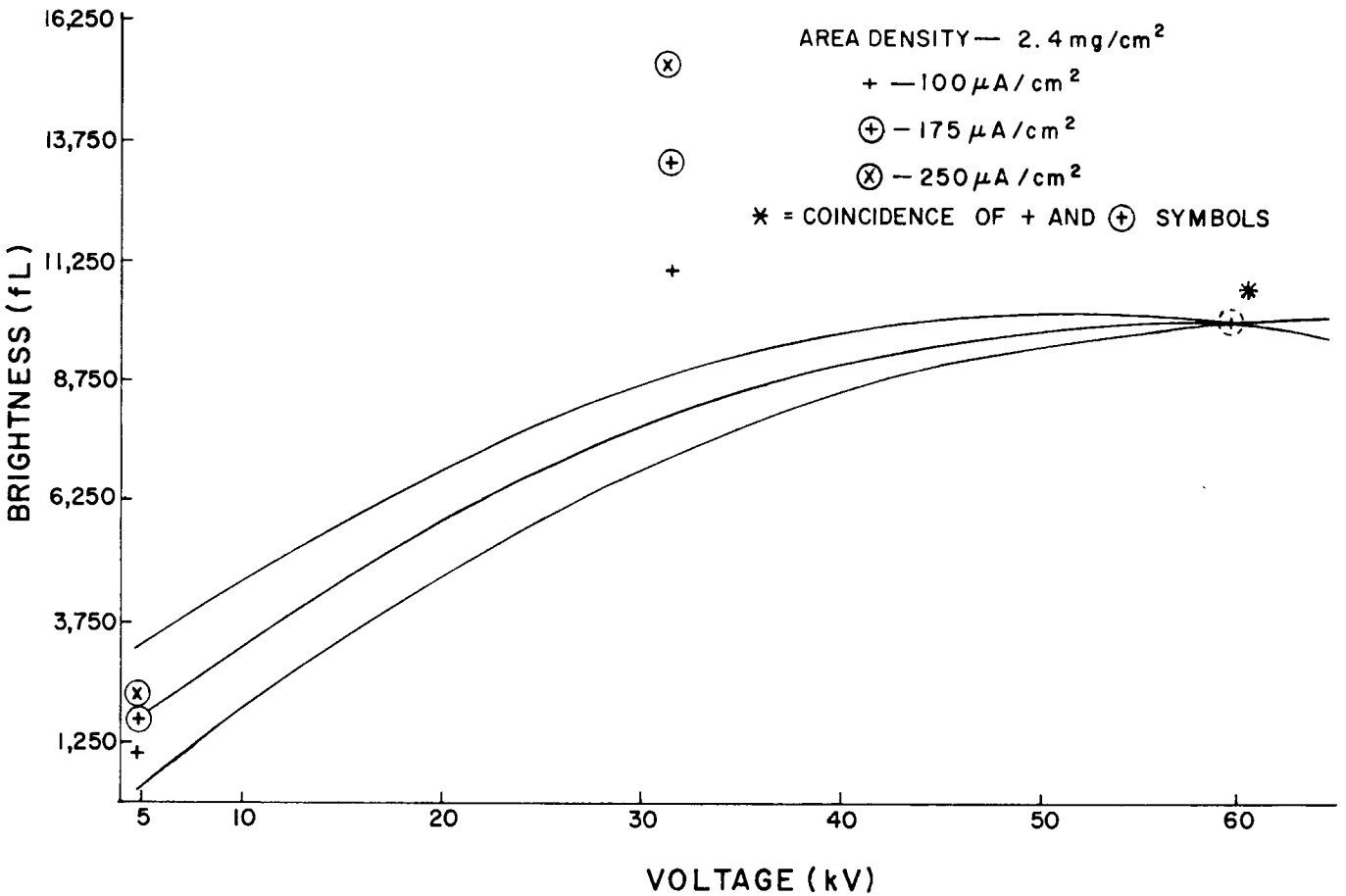

FIG 8 EMPIRICAL CURVES WITH EXPERIMENTAL DATA FOR P24 PHOSPHOR

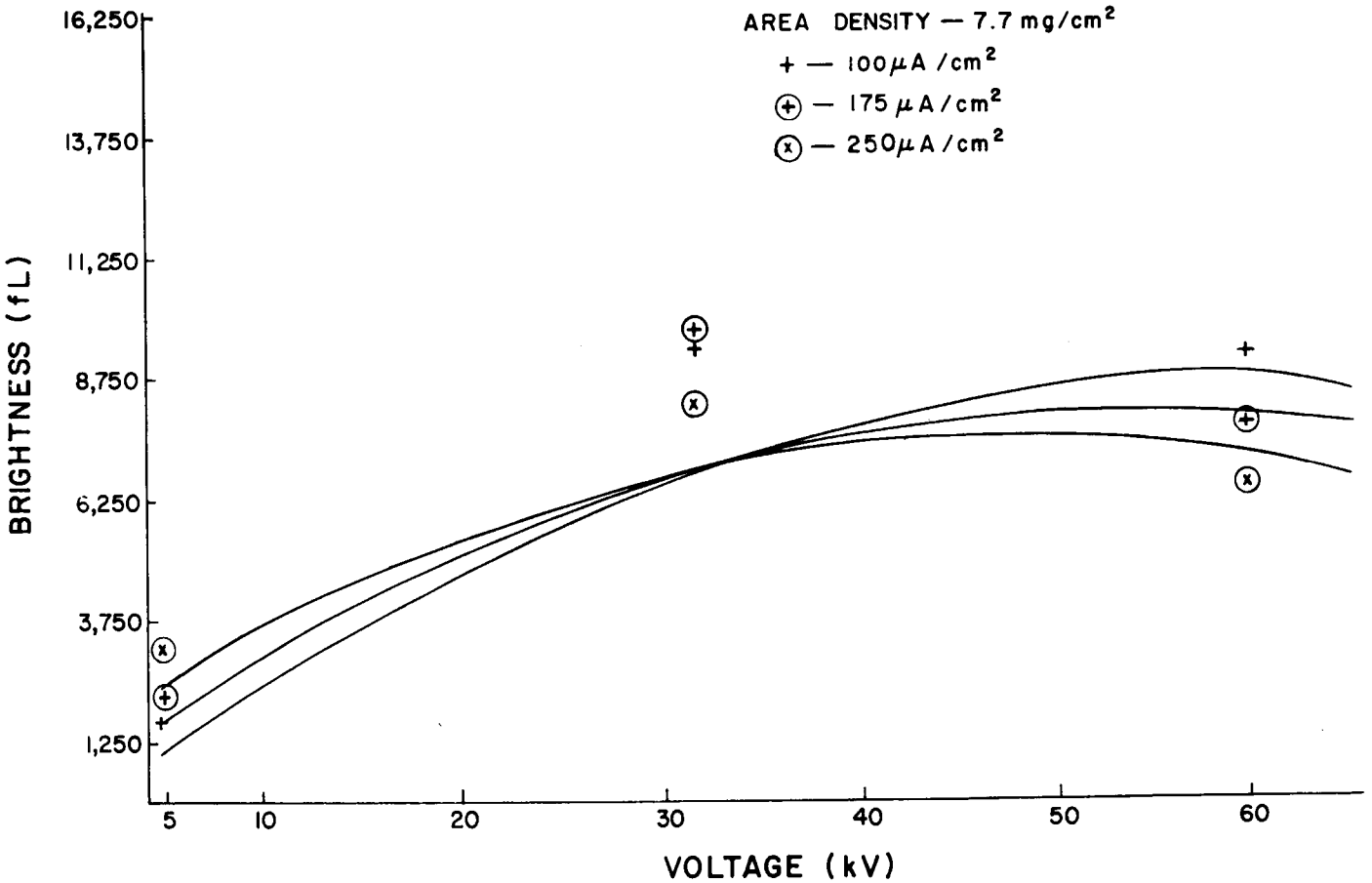

FIG 9 EMPIRICAL CURVES WITH EXPERIMENTAL DATA FOR P24 PHOSPHOR

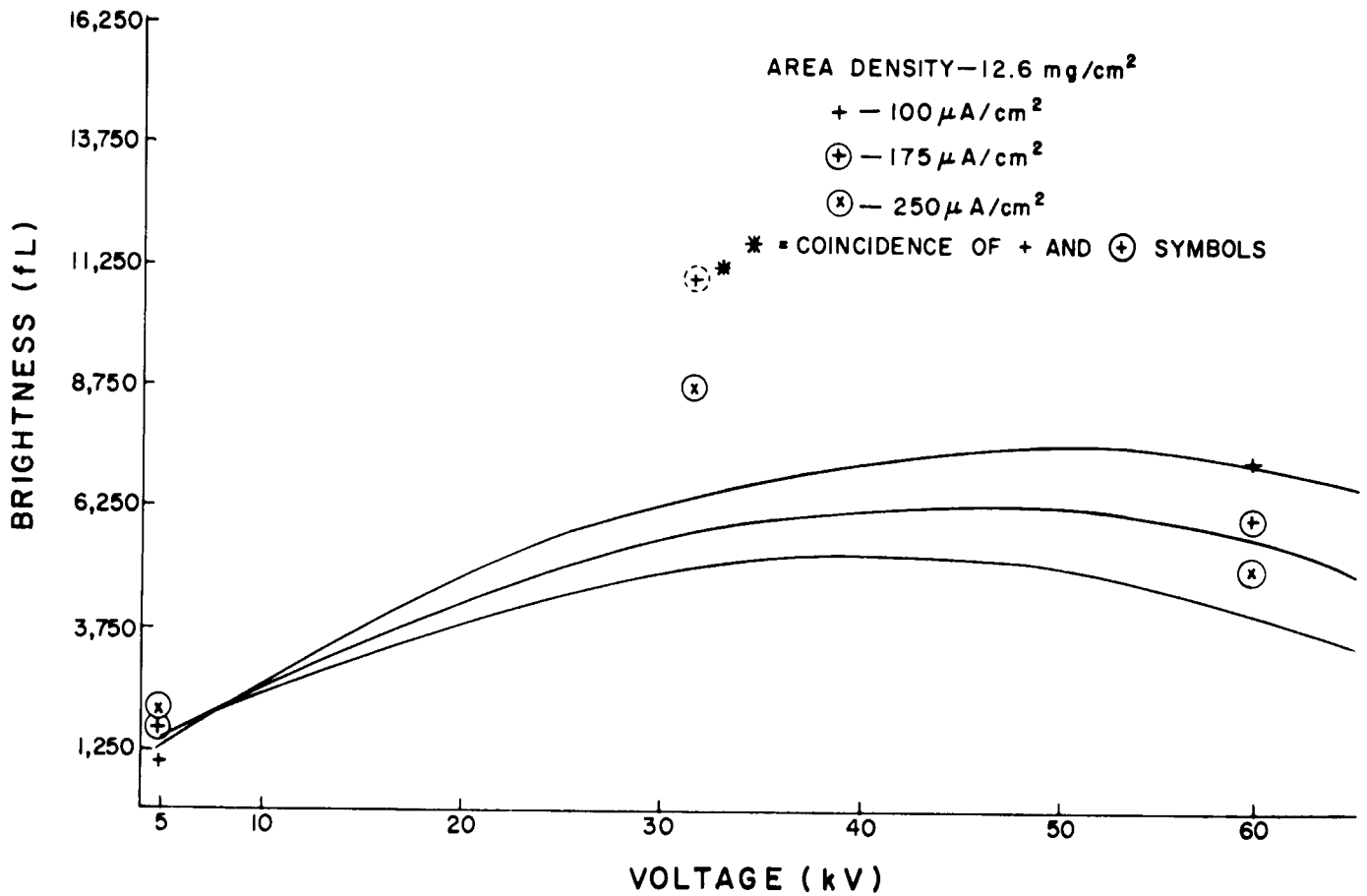

FIG 10 EMPIRICAL CURVES WITH EXPERIMENTAL DATA FOR P24 PHOSPHOR

Figures 11 and 12 show the response or contour surfaces of brightness for P2 and P24 phosphors where the surfaces are drawn in the immediate neighborhood of the actual data points. Again, the points for the contour surfaces were obtained from the general functional relationship for brightness with the assistance of a computer. Since the contour surface for P2 faces outwardly, larger brightness responses may be achieved with higher levels of all the factors. The contour surface for P24 faces inwardly indicating that maximum brightness yield can be obtained with certain combinations of levels.

The existence of surface ridges is of great practical importance. The contour surfaces of P2 phosphor indicate the existence of a stationary ridge in a region outside of where the experimental data were collected. The contour surfaces of P24 phosphor indicate the possibility of having a rising ridge normal to the vertex of each surface cutting through the contour surfaces in a clockwise direction. The significance of a stationary ridge is that one will be able to choose from several combinations of levels that will provide the same optimum light yields from a phosphor. The presence of a rising ridge for P24 phosphor suggests that larger brightness response may be found if one follows the rising ridge.

CONCLUSIONS

Mathematical expressions for cathodoluminescence of P2 and P24 phosphors in regions near optimum light yields, were prepared from empirical techniques. Empirical curves and actual data points were visibly compared to determine the degree of goodness of fit for the derived expressions. Contour surfaces of brightness were generated from the functional expressions. All this was done with the aid of a computer.

The Yates method of analyzing the 3^3 factorial design showed that factors of voltage, beam current density, and phosphor area density are not independent. This would give credence to certain theories of cathodoluminescence (such as the presence of a charge region in phosphors) which would explain certain interaction effects. Any explanation of the mechanism of cathodoluminescence requires some consideration of interaction effects when phosphors are bombarded with a beam of high-energy electrons.

The existence of a stationary ridge suggests that one may operate at several conditions with the same optimum light output. The location of a possible stationary ridge for P2 phosphor is in a region outside of where measurements were made. For P24 phosphor, a rising ridge cuts through the contour surface in a clockwise direction indicating that larger light output may be obtained if one follows the rising ridge.

The 3^3 factorial experimental design provided the required information on the effects of each factor and effects between the factors with a good degree of precision and with a minimum of effort. Since joint effects are

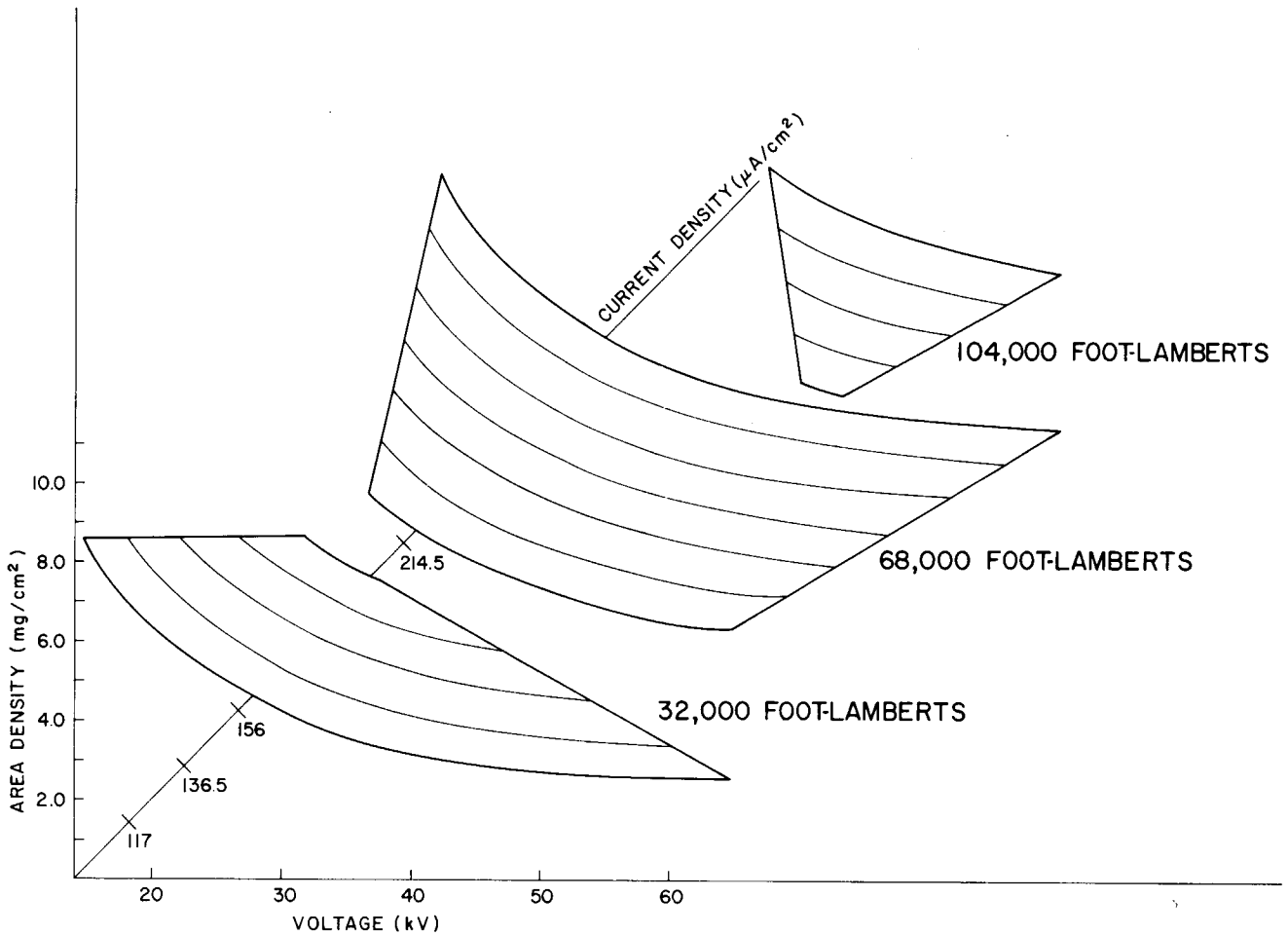

FIG 11 CONTOUR SURFACES FOR P2 PHOSPHOR

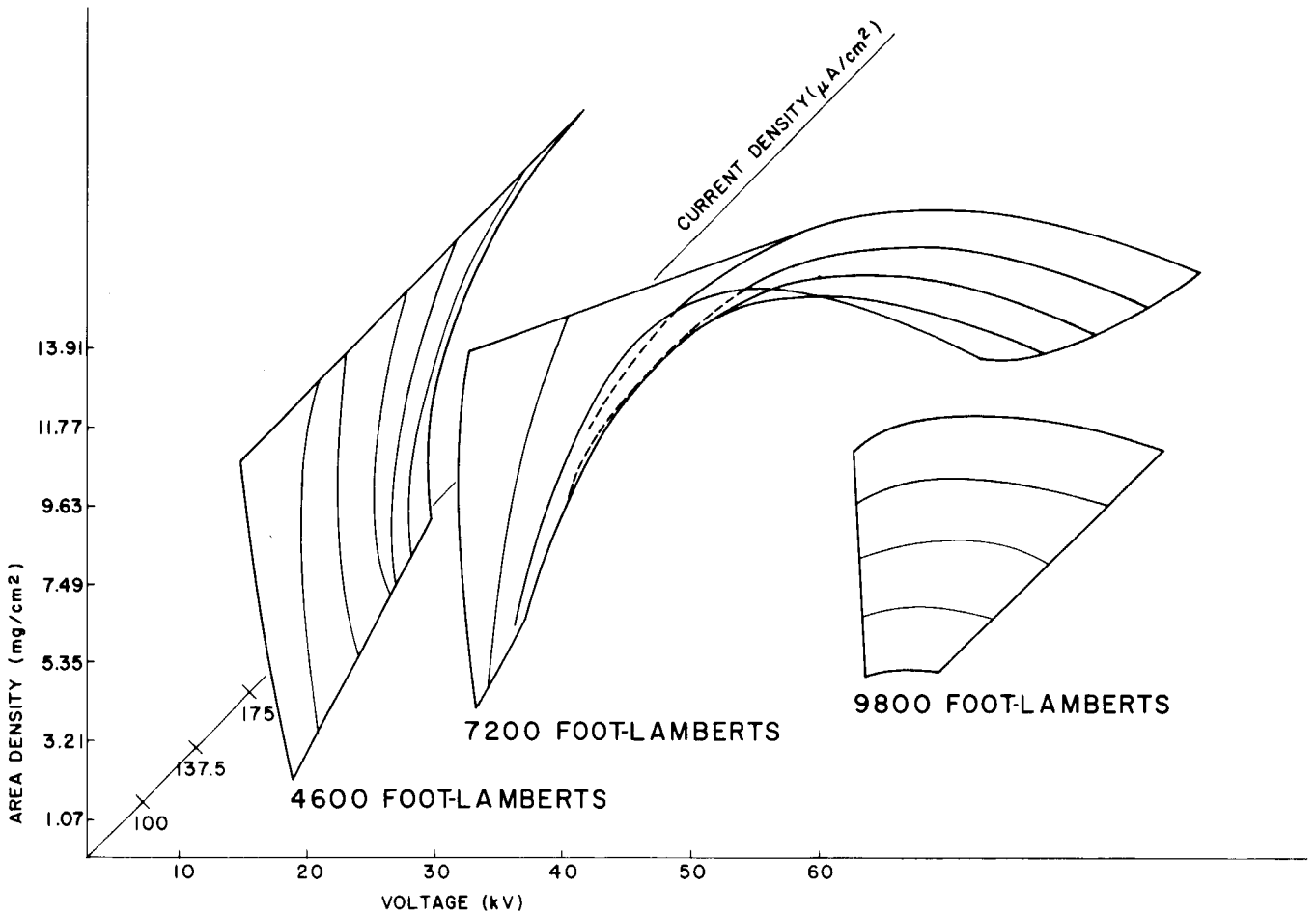

FIG 12 CONTOUR SURFACES FOR P24 PHOSPHOR

significant in the high-energy bombardment of phosphors, there may be phosphors where three levels may not suffice when there are a large number of interactions. Additional experimentation with different phosphors is needed to obtain a better understanding of cathodoluminescence for utilization in developing electronic devices.

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EXPERIMENTAL COMPARISON OF OPERATIONAL TECHNIQUES FOR A SEMIAUTOMATIC FLIGHT OPERATIONS CENTER

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ABSTRACT

During the experimental evaluation of a semi-automatic flight operations center two methods of operation were compared. In one method, active mode, the controller was provided with a display of the complete air traffic situation and was allowed to anticipate and handle problems before being alerted to the problem. In the other method, passive mode, only problem situations were displayed in answer to an alert which the controller received. The statistical comparisons performed and the results obtained are presented. In particular, the analysis of the individual workload items made to gain insight into the unexpected relationship discovered between flight density and total controller workload is discussed.

INTRODUCTION

SAFOC was tested and evaluated at the Federal Aviation Agency (FAA), National Aviation Facilities Experimental Center (NAFEC), Atlantic City, N. J., by the U. S. Army Electronics Command (ECOM), Avionics Laboratory (AMSEL-VL-G), Fort Monmouth, N. J. The evaluation testing of the SAFOC was performed in two phases designated Phase I and Phase II.

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General System Description

Included within the SAFOC system are the following subsystems:

1. Data processing subsystem
2. Radar processing subsystem
3. Display subsystem
4. Manual backup subsystem

The SAFOC provides an air traffic regulation service by collecting, analyzing, and disseminating the information necessary to regulate the movement of Army aircraft under instrument flight rules (IFR) and monitor the movement of cooperating aircraft under visual flight rules (VFR). The air controller(s) perform the air traffic regulation function using data processing and display equipment to provide the following capabilities:

a. Flight Data Processing

    1. Flight data entry-air filed, ground filed
    2. Flight plan clearance
    3. Flight plan activation
    4. Aircraft position determination
    5. Conflict prediction
    6. Collision avoidance
    7. Accomoodating flight plan changes
    8. Flight plan deactivation

b. Flight Following
c. Flight Handoff
d. Identification Assistance
e. Emergency Assistance
f. Air/Ground Coordination
g. Ground/Ground Coordination

SAFOC provides five methods of flight tracking which, in order of their assigned priority, are:

a. Data Link (auto-tracking only)
b. Radar Beacon (auto-tracking)
c. Radar Beacon (rate-aided manual tracking)
d. Radar Skin Return (rate-aided manual tracking only)
e. Plan Follow

Test Configuration

Figure 1 shows the SAFOC test operations and information flow diagram. As illustrated, the scenario generator program, prior to actual testing, generates scenarios and scripts based on random processes. At the time of testing, the scripts are given to the flight simulator pilots, who keep comprehensive logs of all actions.

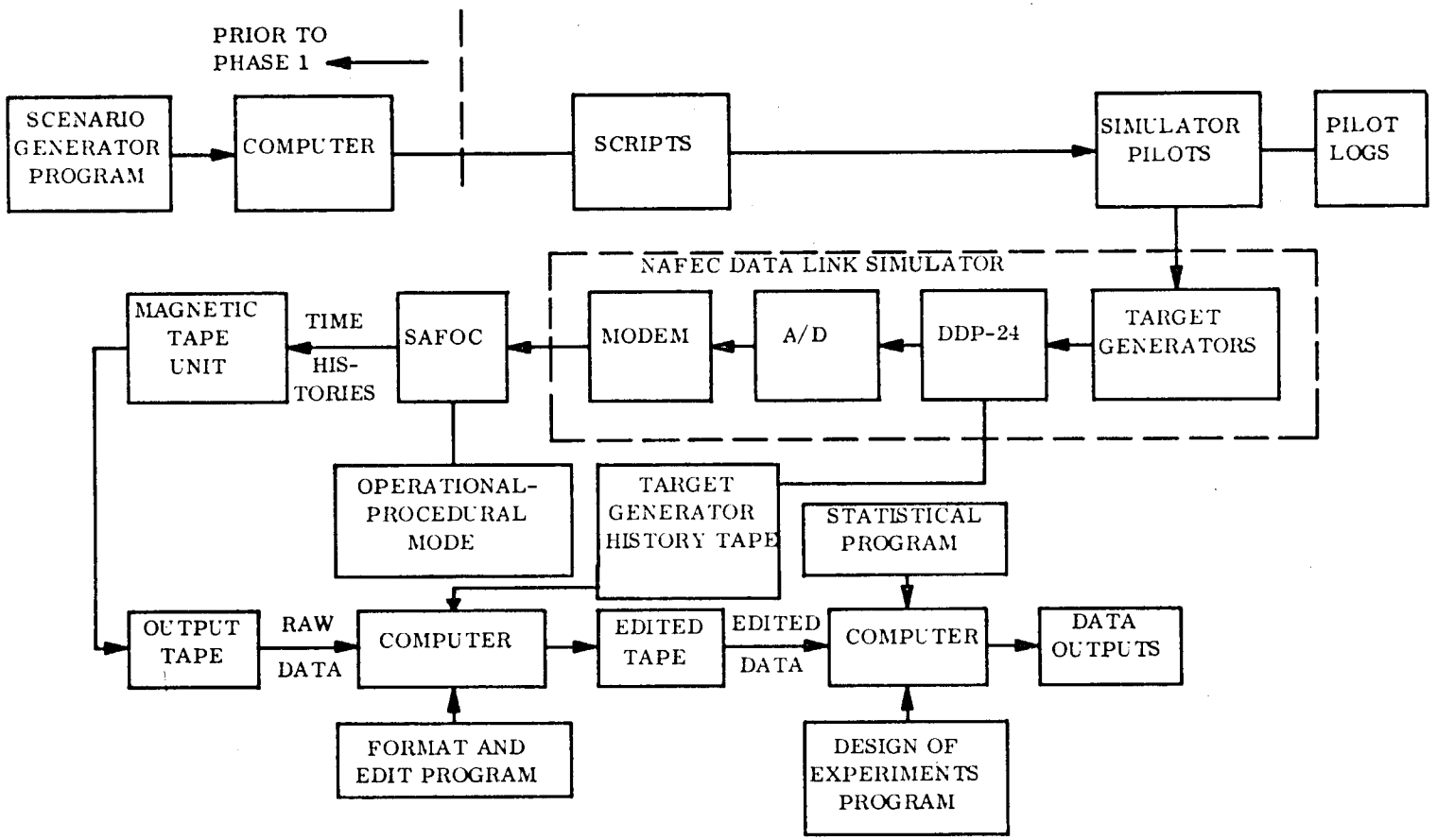

Figure 1. SAFOC Test Operations and Information Flow Diagram

The pilots simulate actual flights using target generators which are part of NAFEC's data link simulation.

Using a pre-determined operational mode, SAFOC controls the simulated flights and produces exhaustive time histories on magnetic tape. These histories include all actions performed by the equipment or by the controller.

The raw output data tape and the target generator history tape are processed using a series of formatting and editing programs.

The processed tapes are then operated upon using statistical program in accordance with the design of experiments, providing the desired data outputs to be described later.

Input Test Parameters

Two methods of operating the SAFOC System are defined as follows:

a. An active mode where all flights are displayed to the operators.
b. A passive mode with displays only as a result of alerts to which the operator reacts.

Three levels of traffic were generated to test the two methods. Six controllers operate the SAFOC for each method at each traffic level. Figure 6 shows the interactions of the test inputs.

The order of performance of the test runs was randomized to eliminate the effects of the learning process of the operators in the course of performing the test runs.

Output Test Parameters

This output data consisted of the system performance measures and system effectiveness measures which were obtained by reducing the data generated from test runs on SAFOC.

The following system performance measures were evaluated:

1. Time to perform a service for each service
2. Service rate
3. Waiting time for service

Service time is time between initiation of service and completion of service. The services to be considered were:

1. Typewriter
2. Handoff
3. Activate (Roll Call or Data Link Initialization)
4. Clear flight plan
5. Coordinate

6. Conflict resolution
7. Alert servicing

   (a) Clearance alert
   (b) Conflict 1 alert
   (c) Conflict 2 alert
   (d) Coordinate alert
   (e) Emergency alert
   (f) Flight overdue alert
   (g) Flight plan complete alert
   (h) Flight not active alert
   (i) Poor tracking alert
   (j) Handoff alert

For each service mentioned above, the service rate is determined by dividing the number of times service is performed in a given time span by the time span duration.

Waiting time for service is the time between initiation of a request for service and the initiation of that service.

The system effectiveness measures were used to provide relative rankings of the operational modes and to evaluate relative controller performance. The following measures were chosen because it is believed they represent the characteristics most important to the user:

a. Safety
b. Controller Workload
c. Communications Workload
d. Delays
e. Throughput
f. Capacity
g. Uncontrolled Time

Safety is defined as the number of near misses per aircraft mile flown. This is measured by computing the number of flights separated by less than the minimum distance during a run divided by the total number of flight miles flown during that run. For the purpose of this measure, the run begins when a steady state density is reached.

Controller workload is defined as total time for all flight hooks in a given time span divided by the time span. Controller workload can be used to estimate controller replacement rates, controller scheduling, etc. Since some tasks the controller must perform may be more taxing than others, workload will be measured by task (handoff, conflict resolution, etc.) as well as by total time that work of any kind is done.

Communications workload is defined as the time spent in communications during a run divided by the duration of the run.

Delays are defined as the actual departure time delay from the planned departure time. Delays are believed to be important in a tactical situation. It does little good to get aircraft safely to a particular location if they arrive too late to be of use in some situations. This measure is easily computed by subtracting actual departure times from the planned departure time.

Throughput is defined as the actual number of flights entered during stydy state divided by the number of planned entries in that time period. The number of flights entered in a given time divided by the number of planned entrances measures the ability of the system to obtain a desired throughput rate.

Capacity is defined as the peak flight density safely handled by the system. This is obtained by performing a regression analysis on near misses vs. peak density. System capacity is that peak density where the resultant regression curve first exhibits significant non-zero slope.

Uncontrolled time is defined as the total time of flights within the SAFOC control area without being controlled by SAFOC.

Differences Between Modes

Part of the experimental evaluation of the test bed involved a comparison of two modes of operation, an active mode and a passive mode. In the active mode, all flights were displayed at all times and the controller was allowed to control the traffic as he saw fit. The computer still provided services to alert him of conditions requiring action, but he need not follow the computer recommendations and could anticipate and avoid problems before the computer notified him of the problem. In the passive mode the display was blank until the controller answered a request for service (an alert) or requested display of a particular flight. Upon answering a request for service the flight for which service is required is displayed along with pertinent information about the flight. At most, two flights were displayed at once. If two flights were in conflict both flights appeared on the display. In the passive mode the controller was forced to accept the computer recommendations and could not anticipate problems. In the passive mode the controller only performed services which were requested. Figures 2 and 3 are examples of active and passive mode displays. From Figure 2 it is seen that at high densities the active mode display becomes very confusing because of the overlap of information.

Method of Evaluating Modes

Figure 4 shows the experimental design for the tests. Each controller was tested at each density level operating in each mode for a total of 36 tests from which the performance and effectiveness measures were obtained for evaluation of the modes of operation.

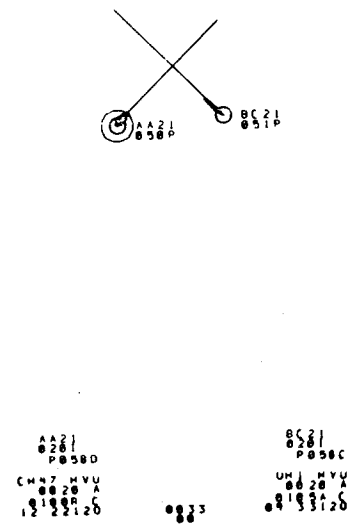

Figure 2. Example of Passive Mode Display

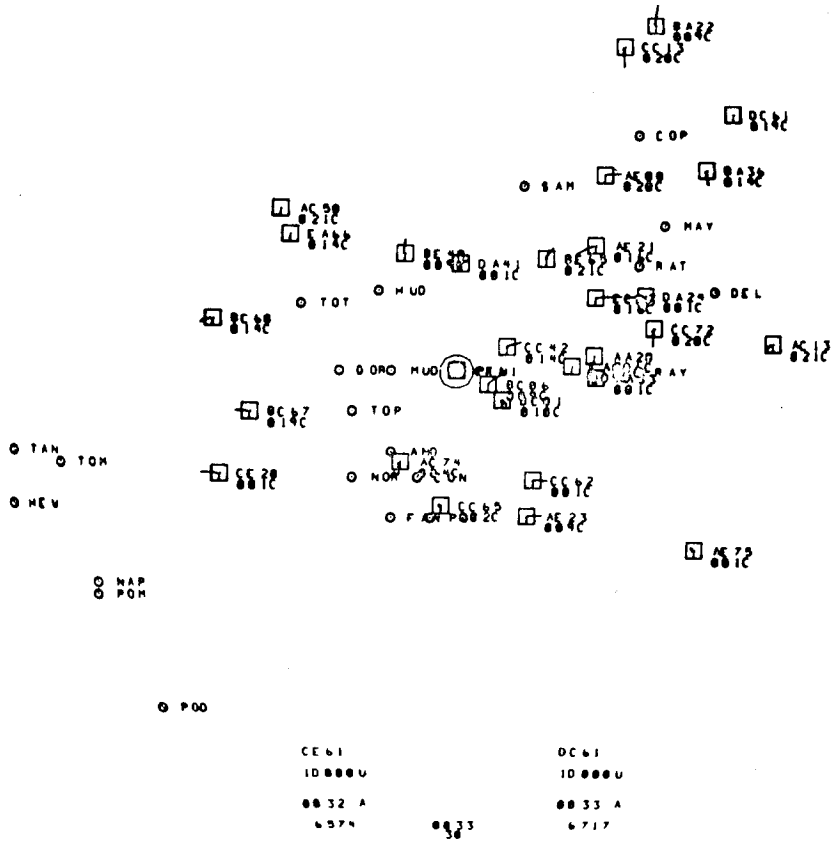

Figure 3. Example of Active Mode Display

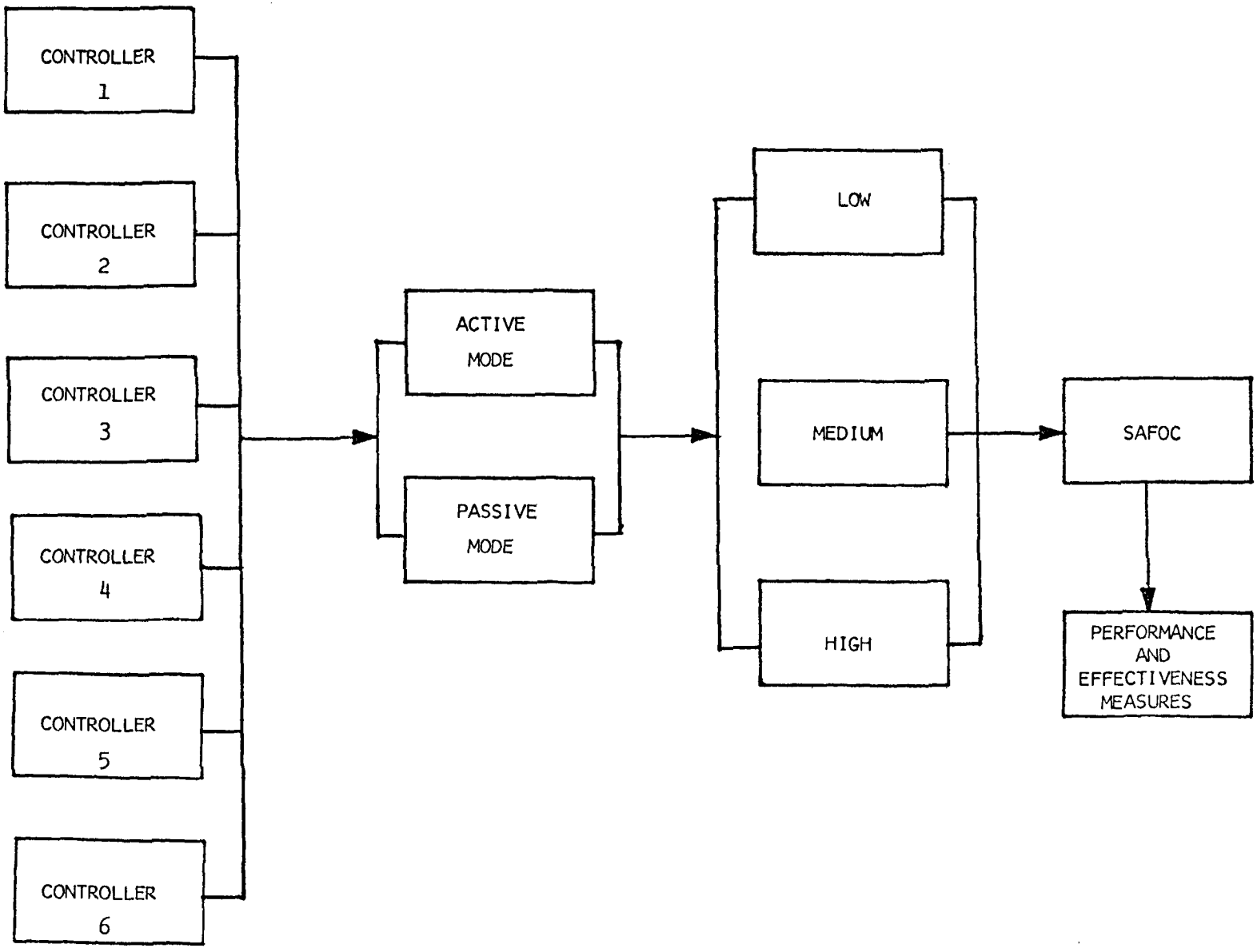

Figure 4. Experimental Design

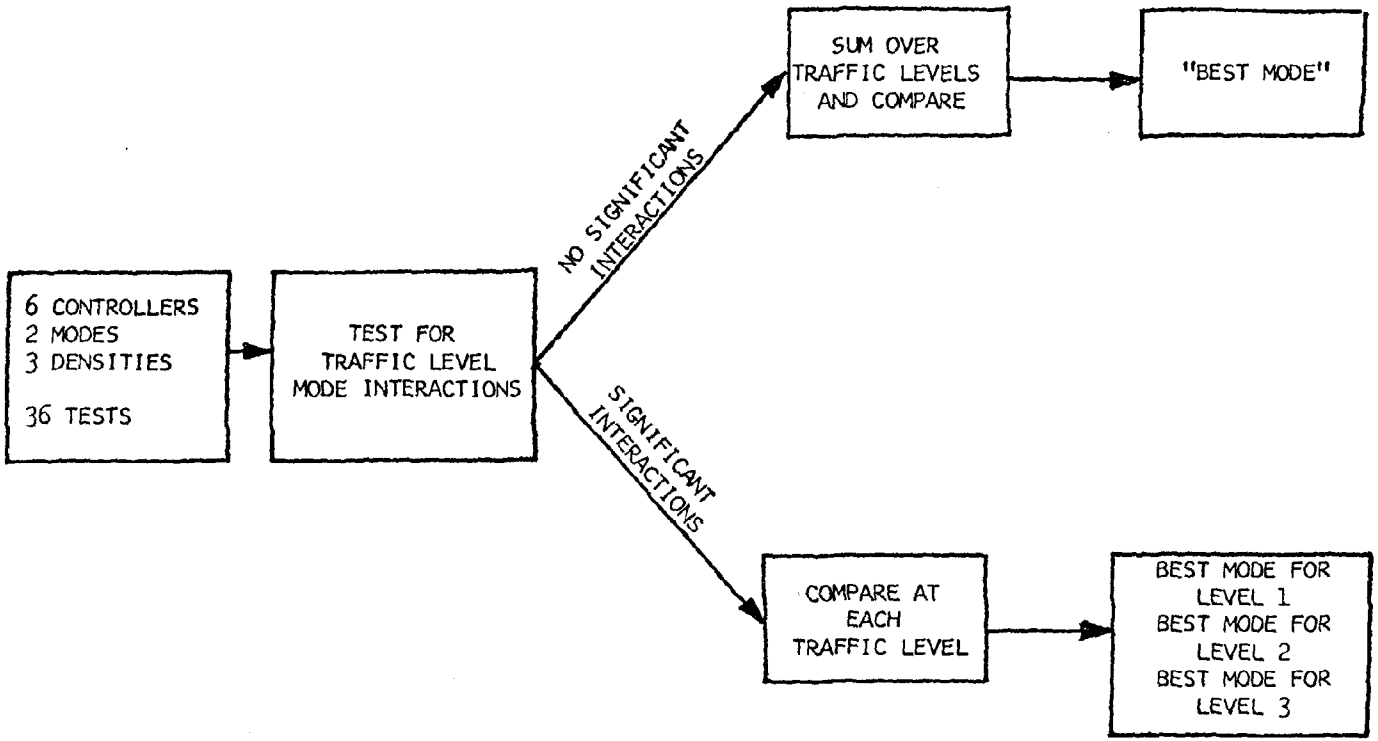

Figure 5. Statistical Testing - Comparison of Modes

The statistical tests for comparison of modes is shown in Figure 5. The first step in the evaluation was to assure that all controllers were operating at equal proficiency levels with retraining where required.

After this assumption is validated, the first test on the 36 runs is a test for interactions between traffic levels and the procedural modes.

If significant interactions are absent, the modes are compared over all traffic levels.

If significant interactions are detected, the modes must be compared at each traffic level.

The total experiment is a mixed model of fixed treatments consisting of three traffic levels and two modes together with random blocks consisting of the six controllers. Another view of the model is that we make six random observations (controllers) on a 6-component vector consisting of the two methods and the three traffic levels. Ranking was performed using the system effectiveness criteria discussed earlier.

The mathematical tests for evaluation of the modes are outlined below.

First the controller performance is tested using Analysis of Variance for a two-way classification as shown in Tables 1 and 2.

$$\text{If } F = \frac{S_2}{S_3} (R-1) < F_{\alpha\,;\,(c-1),\,(R-1)\,(C-1)}$$

accept the null hypothesis of no controller difference.

Then mode interactions are tested as shown on the following page.

Table 1. Effects of Replications

CONTROLLERS

Replications	1	2	•	•	C	Row Sum
1	X_{11}	X_{12}	•	•	X_{1C}	$\sum_{j}^{C} X_{ij} = C\bar{X}_{1.}$
2	X_{21}	X_{22}	•	•	X_{2C}	$\sum_{j}^{C} X_{2j} = C\bar{X}_{2.}$
•	•	•	•	•	•	
•	•	•	•	•	•	
R	X_{R1}	X_{R2}	•	•	X_{RC}	$\sum_{j}^{C} X_{Rj} = C\bar{X}_{R.}$
Column Sum	$R\bar{X}_{.1} = \sum_{i}^{R} X_{i1}$	$\sum_{i}^{R} X_{i2} = R\bar{X}_{.2}$			$R\bar{X}_{.C} = \sum_{i}^{R} X_{iC}$	$\sum_{ij}^{RC} X_{ij} = RC\bar{X}_{..}$

C = Number of controllers
R = Number of replications

Table 2. Analysis of Variance for a Two-Way Classification

SOURCE	SUMS OF SQUARES	DEGREES OF FREEDOM	MEAN SQUARE
Replications	$S_1 = \sum_{i}^{R} (\bar{X}_{1.} - \bar{X}_{..})^2$	R-1	$S_1/(R-1)$
Teams	$S_2 = \sum_{i}^{C} (\bar{X}_{.j} - \bar{X}_{..})^2$	C-1	$S_2/(C-1)$
Error	$S_3 = \sum_{i}^{R} \sum_{j}^{C} (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2$	(R-1) (C-1)	$S_3/(R-1)$ (C-1)
Totals	$S_4 = \sum_{i}^{R} \sum_{j}^{C} (X_{ij} - \bar{X}_{..})^2$	RC-1	---

Mode Interactions with Traffic Level

Assume: $X_{ijh} = \mu_{jh} + \epsilon_{ijh}$; $i = 1,2,.., N_j$; $j = 1,2,.., k$; $h = 1,2,...p$

or $X = A_\mu + \epsilon$ in matrix form

X_{ijh} = i^{th} observation of effectiveness measure for j^{th} mode at traffic level h

μ_{jh} = mean value of effectiveness measure for j^{th} mode at traffic level h

ϵ_{ijh} = error in i^{th} observation for j^{th} mode at traffic level h

Test null hypothesis

H_o: $\mu_{11} - \mu_{12} = \mu_{21} - \mu_{22}$ or H_o: $C\mu L = 0$ in matrix form

$\mu_{12} - \mu_{13} = \mu_{22} - \mu_{23}$

Compute:

$$H = L^T X^T A(A^T A)^{-1} C^T \left[C (A^T A)^{-1} C^T\right]^{-1} C(ATA)^{-1} A^T XL$$

$$E = L^T X^T \left[I - A(A^T A)^{-1} A^T\right] XL$$

Maximum characteristic root of HE^{-1} is distributed as Heck Maximum Characteristic Root Distribution with parameter

$$S = \text{Min}(k-1, p-1),\ m = \frac{|k-p| - 1}{2},\ n = \frac{\Sigma N_j - k - p}{2}$$

For test $N_j = 6$ (allj), k=2, and p=3

For test situation a simple procedure is available

The single non-zero root of HE^{-1} is $C = t_r\, HE^{-1}$

where t_r = trace of matrix

$F = \left[\frac{n+1}{m+1} C\right]$ is distributed as F distribution with 2m+2 and 2n+2 degrees of freedom

If $F \leq F_{\alpha, 2m+2, 2n+2}$ accept hypothesis of no interaction at some significance level α .

Finally the modes are compared as described below.

A. No Traffic Level Interactions

Hypothesis: H_o: $\mu_{1.} - \mu_{2.} = 0$

Compute: $\bar{X}_{.1.}$ and $\bar{X}_{.2.}$

where $\bar{X}_{.j.} = \sum_{h=1}^{3} \sum_{i=1}^{N} \frac{X_{ijh}}{3N}$, $j = 1,2$

and $\mu_{j.} = \frac{1}{3} \sum_{h=1}^{3} \mu_{jh}$

Compute mean difference $\bar{D} = \bar{X}_{.1.} - \bar{X}_{.2.}$

and standard error

$$S_{\bar{D}} = \frac{1}{\sqrt{N}} \sqrt{\frac{1}{N-1} \left[\sum_{i=1}^{N} (d_i)^2 - \frac{1}{N} \left(\sum_{i=1}^{N} d_i \right)^2 \right]}$$

where $d_i = \frac{1}{3} \sum_{h=1}^{3} (X_{i1h} - X_{i2h})$

If the $1 - \alpha$ confidence interval

$$\bar{D} - (t_{\alpha/2;\, n-1})\, S_{\bar{D}} \leq \mu_{1.} - \mu_{2.} \leq \bar{D} + (t_{\frac{\alpha}{2};\, n-1})\, S_{\bar{D}}$$

contains zero accept the hypothesis

B. Significant Traffic Level Interactions

Same as in A except

$\bar{D}_h = \bar{X}_{.1h} - \bar{X}_{.2h}$, $h = 1, 2, 3$ replaces $\bar{D}$

where $\bar{X}_{.jh} = \frac{1}{n} \sum_{1}^{n} X_{ijh}$

and $d_{ih} = X_{i1h} - X_{i2h}$ replaces d_i

Results

The confidence intervals computed for the effectiveness measures used for comparing the modes of operation are given in Table 3.

No significant interactions between mode and density were found for the Safety measure. All other effectiveness measures exhibited significant interactions.

Significant differences between modes were found only for the Safety measure and the Workload measure. Active mode is safer than passive mode at all densities while requiring more workload than passive mode at low and medium densities. At high densities the active mode workload was less than the passive mode, however this difference was not statistically significant at the .2 level.

Table 4 summarizes the average active and passive mode effectiveness measures obtained at each density. From this table it is seen that the active mode workload decreased by 4.3 minutes per hour when the density increased from medium to high. That is, even though there were more flights to be handled, the controller spent less time servicing the flights. Similar relationships are also indicated for communication workload. This unexpected workload decrease is even more evident in Figure 6 which shows the results of a polynomial regression analysis of workload vs. density. Polynomial regression analyses with analysis of variance to determine if the regression fit is significant were performed for all the effectiveness measures. However, only the workload measures as shown in Figure 6 indicated an unexpected trend. One would expect workload to reach some saturation level and remain at that level rather than to decrease with density.

No significant fit could be found for Safety vs. density indicating that Safety is independent of density. It is known that as density increases the probability of a near miss increases, so that the Safety should deteriorate as density increases.

In order to determine reasons for the observed density effects and mode differences, more detailed investigations of the test data was made.

Investigation of Safety Differences

Before attributing the observed Safety differences to differences between the modes of operation it is necessary to assure that no uncontrolled parameters effected the comparisons between modes.

Since the same flight paths were flown for both active and passive mode tests the differences could not be attributed to the test scenarios used.

Since the same controllers were used for both modes the difference cannot be attributed to controllers.

Since the order of testing was completely randomized according to both mode and density, learning effects upon the comparison were minimized.

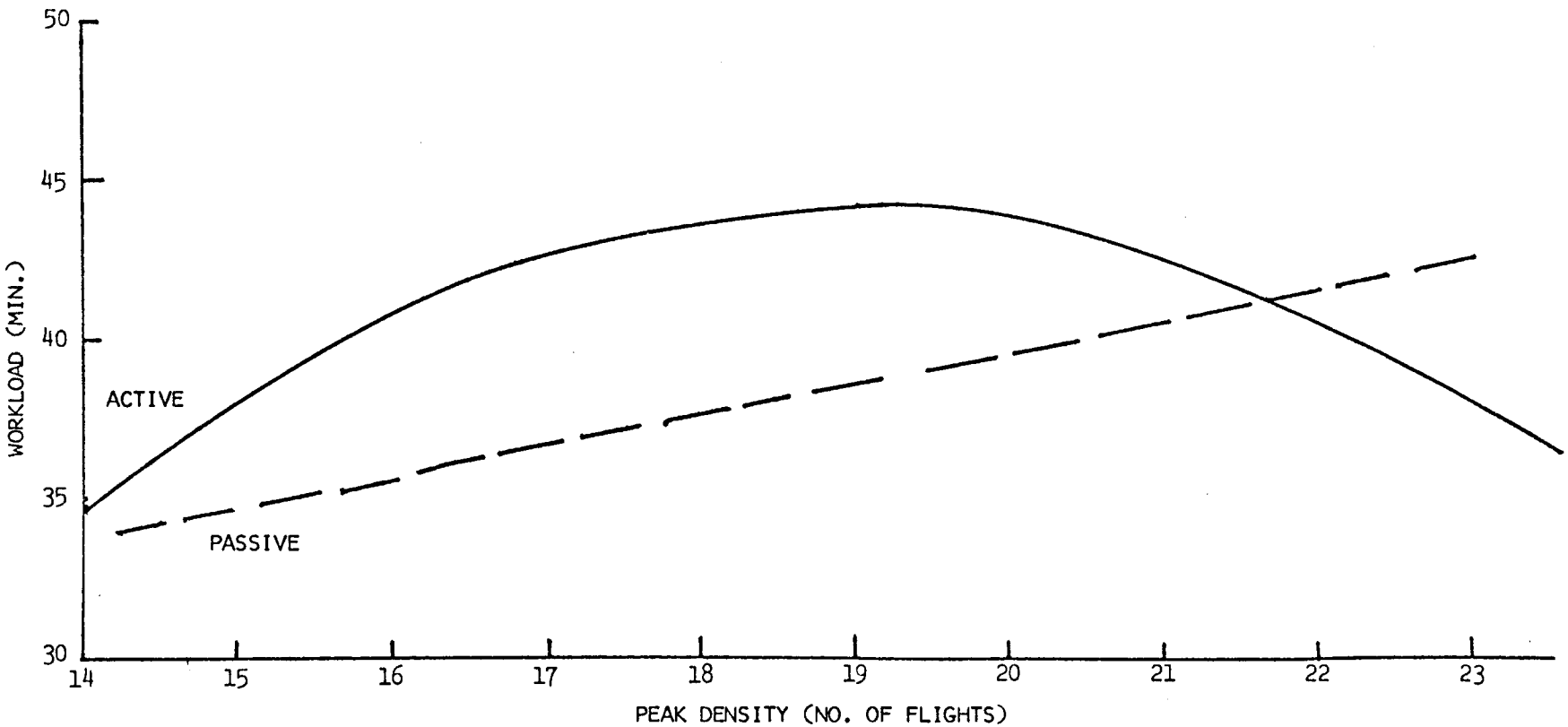

Figure 6. Regression Curves, Workload vs. Peak Density
(Per Controller for Sixty Minute Test)

TABLE 3

CONFIDENCE INTERVALS OF DIFFERENCES BETWEEN

MEANS OF EFFECTIVENESS MEASURES (ACTIVE-PASSIVE)

(Phase I Tests)

Effectiveness Measure	Traffic Density	Lower Limit of Interval	Upper Limit of Interval	Confidence Level	Significant Difference
Safety	Combined*	-.00095	-.00005	90%	Yes
Workload	Low	1.35	11.35	80%	Yes
	Med	0.17	8.83	80%	Yes
	High	-4.02	0.34	80%	No
Delay	Low	-.89	6.35	80%	No
	Med	-9.33	0.67	80%	No
	High	-6.10	8.10	80%	No
Uncontrolled Time	Low	-.51	2.67	80%	No
	Med	No uncontrolled time accumulated			
	High	-8.663	7.137	80%	No
Throughput	Low	-.0815	.0449	80%	No
	Med	-.1187	.103	80%	No
	High	-.0147	.0273	80%	No
Communications Workload	Low	-.483	1.317	80%	No
	Med	-.96	1.146	80%	No
	High	-.346	.816	80%	No

* Combined over three traffic densities

TABLE 4

SUMMARY OF ACTIVE AND PASSIVE

MODE EFFECTIVENESS COMPARISONS

Measure	Density Level	Average Active Mode	Average Passive Mode	Average Active Passive Mode
Safety*	Low	.0002	.000617	-.000417
Safety*	Med	.000483	.000867	-.000383
Safety*	High	.000283	.000983	-.000700
Workload	Low	37.1 min.	30.7 min.	6.35 min.
Workload	Med	42.9 min.	38.4 min.	4.50 min.
Workload	High	38.6 min.	40.4 min.	-1.80 min.
Delay	Low	6.17 min.	3.33 min.	2.83 min.
Delay	Med	6.67 min.	11.00 min.	-4.33 min.
Delay	High	12.50 min.	11.50 min.	1.00 min.
Uncontrolled Time	Low	1.84 min.	0.760 min.	1.08 min.
Uncontrolled Time	Med	0.00 min.	0.00 min.	0.00 min.
Uncontrolled Time	High	7.04 min.	7.80 min.	-0.76 min.
Throughput	Low	1.02	1.04	-.02
Throughput	Med	1.00	1.01	-.01
Throughput	High	.967	.960	.007
Communications	Low	5.29 min.	4.88 min.	0.41 min.
Communications	Med	6.69 min.	6.21 min.	0.48 min.
Communications	High	5.98 min.	5.75 min.	0.23 min.

* Safety is measured in near misses per mile flown

Low = 14 - 16 flights peak

Med = 17 - 19 flights peak

High = 20 - 23 flights peak

The only uncontrolled factor which may have influenced the comparison was poor tracking quality. Because of test equipment problems, excessive poor tracking alerts occurred on several tests. It was hypothesized that these unpredictable poor tracking problems may have caused the observed difference in the Safety measure. If a flight is not being properly tracked, one would expect that Safety to deteriorate. Perhaps the poor tracking problem occurred more frequently on passive mode tests. A t-test for significant differences in the number of poor tracking alerts between active and passive mode tests, however, indicated no significant differences. Also a correlation between safety and the number of poor tracking alerts indicated very low correlation (.098). Thus, it is concluded that the non-controlled tracking problems did not affect the mode comparisons for the Safety measure.

The observed difference must be caused by inherent differences between modes. The only difference between modes, however, is in the information displayed. In active mode the entire air traffic is displayed, while in passive mode only problems to which the controller has been alerted are displayed. It was, therefore, hypothesized that active mode is safer than passive mode because anticipation of conflicts is possible in active mode. If this hypothesis were correct one would expect less conflict alerts in active mode.

Table 5 indicates the differences in the numbers of conflict alerts at the various densities. Although these differences are not statistically significant at the .2 level, there is consistently fewer conflict alerts for active mode operation at all densities supporting the hypothesis that anticipation of conflicts is occurring in active mode.

Detailed analyses of the differences in times to perform the various services also indicated that significantly longer times were required in active mode to perform those services during which anticipation of conflicts might take place.

A detailed analysis of each near miss that occurred in the 36 tests indicated that over 70% were caused by insufficient warning time or insufficient conflict alert durations. This points out the deficiencies in the conflict prediction algorithms used and also explains why no density and Safety relationship was found. Evidently the deficiencies in the algorithms used obscured any density effects.

Investigation of Workload Differences

In order to investigate the reasons for the observed differences in workload, detailed studies were made of the differences in services and service times between the two modes.

Table 6 indicates that there were less services performed in active mode than in passive mode but Table 7 indicates that the average service time was higher, resulting in more workload in active mode.

Table 8 shows a comparison of the incomplete events for the two modes of operation. An incomplete event is a request for service which goes unanswered.

TABLE 5

AVERAGE NUMBERS OF NEAR MISSES AND CONFLICT ALERTS PER RUN

BY WHICH PASSIVE MODE EXCEEDS ACTIVE MODE

(Phase I Tests)

Safety Measures	Traffic Density		
	Low	Medium	High
T = 0	.33	1.0	1.5
Conflict 1	6.33	8.83	45.83
Conflict 2	9.83	8.83	13.33
Conflict 1/2	.50	0	4.5

T = 0 Time to go until crash is zero seconds

Conflict 1: Time to go until crash is between 0 minutes and 1 minute

Conflict 2: Time to go until crash is between 1 minute and 3 minutes

Conflict 1/2: Data is insufficient to determine whether alert is Conflict 1 or Conflict 2.

TABLE 6

SERVICES PER RUN COMPARISON

Density	Average Active	Average Passive	Average Diff. Active-Passive	Significant Diff. at .2 Level
Low	173.9	173.9	0.0	No
Med	191.7	206.0	-14.3	No
High	202.3	220.1	-17.8	No

TABLE 7

AVERAGE SERVICE TIME COMPARISONS

Density	Average Difference Active Minus Passive (Sec/Hook)		95% Confidence Interval (Sec/Hook)
Low		+ 1.38	+ 1.38 ± 1.38
Med	+ 2.52	+ 2.52	+ 2.52 ± 1.32
High		+ 0.96	+ 0.96 ± 1.26

TABLE 8

INCOMPLETE EVENTS PER RUN COMPARISON

Density	Average Active	Average Passive	Average Diff. Active-Passive	Significant Diff. at .2 Level
Low	7.2	17.7	-10.5	Yes
Med	12.0	9.5	2.5	No
High	27.0	12.5	14.5	No

If significance level relaxed to 0.3 the difference at high density becomes significant

Low = 14-16 flights peak
Med = 17-19 flights peak
High = 20-23 flights peak

From this table it is seen that in active mode of operation the number of incomplete events increased considerably when the density increased from medium to high. This explains at least part of the decrease in workload observed in active mode.

Evidently requesting more than the controllers are able to handle results in frustration and performance below their maximum capabilities. When the number of requests for service became too great the controller did not continue to work at peak levels answering all requests that they could, but actually ignored requests that could have been answered if the peak level of workload had been maintained.

The differences in the numbers of incomplete events between active and passive modes only explains 60% of the workload difference. Evidently there is another factor present. To investigate this the individual service times for each service were analysed. Tables 9 and 10 indicate those service times which exhibited statistically significant differences between modes and between density levels..

Clear and Clear/Coordination combination service and Handoff/Roll Call and Data Link Initialization services are those services during which anticipation of conflicts would occur. The differences in these service times between active and passive modes is attributed to the extra anticipation of conflicts that is performed in active mode. However, a speed-up in these services was noticed at high density in active mode as indicated in Table 7. This speed-up indicates that the anticipation of conflicts is being dropped at the high density. There is a general speed-up in service times as density increases indicating that the controllers are working faster and dropping extra services that could be performed in active mode. The exceptions to this speed-up were console alert, monitor and Aux data services. The console alert service occurs when a conflict alert is automatically reset while the controller has the flight hooked. Evidently the controller spends more time to determine if there really is a conflict situation on the confusing display at higher densities. This cannot be done in passive mode, however, because the entire air traffic is not displayed. Monitor service is a combination of all services which did not fit into a pre-defined category. These services generally took longer in active mode as the density increased because of the increased confusion in the active mode display. The Aux data service exhibited longer service times in passive mode than in active mode at the low and medium densities, but active mode exhibited longer times at high density. Aux data service consists of displaying the flight plan and flight plan information to determine where a flight is going. Evidently at lower densities in passive mode the controller spends extra time trying to remember the relationships of other flights not displayed to the flight plan. At high density this is dropped. Because of the more confusing display in active mode at high density the retrieval of flight plan information from the display takes considerably longer.

Conclusions

From the analyses performed the following conclusions were reached:

1. Active mode is safer than passive mode because it allows the controller to anticipate conflicts and avoid them.

TABLE 9

SIGNIFICANT DIFFERENCES IN

SERVICE TIMES WITH DENSITY

ACTIVE MODE			PASSIVE MODE		
Service	Densities*	Difference (Sec)	Service	Densities*	Difference (Sec)
Handoff	L-M	7.3	Coordinate	L-M, M-H	2.5, -2
Poor Track	L-M, L-H	5.4, 5	Conflict 2	L-M	4
Console Alert	L-M	-4	Clearance	L-M	-2
Clearance	M-H, L-H	1.3, 1.3	Conflict 1/2	L-M	3.3
Auxiliary Data	M-H, L-H	-8, -6	Handoff/Roll Call	L-M, M-H	3, -3
Handoff/Roll Call	M-H	2	Auxiliary Data	M-H, L-H	8, 7.5
Monitor	M-H, L-H	-6, -4	Data Link Initialize	L-H	-6.7
Conflict 2	L-H	4.3			

* L-M = low minus medium
L-H = low minus high
M-H = medium minus high

TABLE 10

SIGNIFICANT DIFFERENCES IN

SERVICE TIMES WITH MODE

(Active Minus Passive in Sec.)

Low Density		Medium Density		High Density	
Clearance	2	Conflict 1	2	Conflict 1	2
Poor Track	3	Coord.	3	Coord.	3
Data Link Initialize	5.5	F/P Complete	3	Conflict 2	-4.2
Auxiliary Data	-5.5	Poor Track	2	Clear/Coord.	2
Monitor	5	Clear/Coord.	3	Auxiliary Data	8
		Auxiliary Data	-8	Handoff Out	5
		Console Alert	4	Monitor	6
		Handoff/Roll Call	5		

2. Because of additional time spent anticipating problems and retrieving information from a cluttered display, active mode requires more controller workload.

3. Requesting more work than the controllers are able to handle results in frustration and performance far below maximum capabilities.

4. The best system will be obtained by using passive mode with improved conflict prediction to eliminate the necessity for controller intervention to maintain safety.

The above conclusions were incorporated into recommendations for an optimal system.

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A Review of the Theory and Application of Methods for Comparison of Proportions

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1. Introduction and Summary

Probably the most common quantitative problem in biological research involves the question of the presence or absence of a particular attribute. In its simplest form it reduces to a 2 x 2 table analysis or a test for the difference between two proportions; in a more complex form it may involve the comparison of an array of proportions adjusted for an auxiliary variable.

Many of the proposed methods for treating these problems may be subsumed under the general framework of logistic models (see e.g. Cox [1958, 1970]). The analysis uses the notion of conditioning the distribution of the test statistic on the ancillary statistics, which implies, in most cases the so-called "fixed marginals" analysis of contingency tables. Using this theory, Gart [1971] collated the extensive literature on this subject. This review relied to a large extent on the excellent papers in this area by Birch [1964, 1965] and Armitage [1966]. More recently Gart [1972] has extended the test of interaction for 2 x 2 x t tables by Zelen [1971] to cover the general case of 2 x b x t tables.

In this paper we discuss the various significance tests. The reader is referred to Gart [1970, 1971] for the related questions of point and interval estimation.

2. The General Problem of Testing Treatment Effects

Consider an array of mutually independent binomial variates X_{ij}, based on samples of size, n_{ij}, with parameter, P_{ij}, where i=1, 2,... b; j=1, 2,... t. The logistic model is assumed:

$$P_{ij} = \frac{\exp(\mu + \beta_i + \tau_j)}{1 + \exp(\mu + \beta_i + \tau_j)},$$

for all i and j, where $\Sigma \beta_i = \Sigma \tau_j = 0$ and $Q_{ij} = 1 - P_{ij}$. The β's relate to the rows and may be considered the block effects while the τ's relate to the columns and may be considered the treatment effects. The model assumes no interaction in the usual sense for a higher order contingency table, that is,

$$\frac{P_{ij} Q_{ik}}{Q_{ij} P_{ik}} = e^{\tau_j - \tau_k} \qquad \begin{cases} i=1, 2,... b \\ j \neq k = 1, 2,... t \end{cases}$$

which implies the respective cross product ratios are constant across the rows or blocks.

Consider first the problem of testing no treatment effects, i.e. H_o: $\tau=0$, j=1, 2,... t against H_1: $\tau_j \neq \tau_k$, some $j \neq k = 1, 2,... t$. The likelihood under H_1 is

$$L(\mu, \beta_i, t_j; x_{ij}) = \frac{\exp\{x_{..}\mu + \Sigma_i x_{i.}\beta_i + \Sigma_j x_{.j}\tau_j\}}{\prod_{i,j}(1+e^{\mu+\beta_i+\tau_j})^{n_{ij}}}.$$

We see we have a jointly sufficient set of statistics, $x_{..}$, $x_{i.}$, and x_{ij} and inferences about the τ's should be based on the joint conditional distribution of the $X_{.j}$ with the $x_{i.}$ fixed, which of course implies that $x_{..}$ is fixed (Lehmann [1959]). Under H_o the exact conditional distribution of the X_{ij} is a product of b independent multivariate hypergeometric distributions.

$$h(X_{ij} \mid x_{i.} : \tau_j = 0 = \frac{\prod_{i=1}^{b} \prod_{j=1}^{t} \binom{n_{ij}}{x_{ij}}}{\prod_{i=1}^{b} \binom{n_{i.}}{x_{i.}}}. \tag{2.1}$$

From which we have

$$E(X_{ij} \mid x_{i.} : \tau_j = 0) = e_{ij} = n_{ij} \frac{x_{i.}}{n_{i.}}, \tag{2.2}$$

$$V(X_{ij} \mid x_{i.} : \tau_j = 0) = \frac{n_{ij} x_{i.} (n_{i.} - x_{i.})(n_{i.} - n_{ij})}{n_{i.}^2 (n_{i.} - 1)}, \tag{2.3}$$

and

$$C(X_{ij}, X_{ik} \mid x_{i.} : \tau_j = 0) = \frac{-n_{ij} n_{ik} x_{i.} (n_{i.} - x_{i.})}{n_{i.}^2 (n_{i.} - 1)}, \tag{2.4}$$

for all possible i, $j \neq k$.

For a composite null hypothesis neither the Fisherian nor the Neyman-Pearson theory leads to a unique optimal test statistic for H_o. Two test statistics have been proposed.

Armitage [1966] suggested a chi-square statistic based on the $x_{.j}$'s,

$$X_a^2 = \sum_{j=1}^{t} (x_{.j} - e_{.j})^2 \left(\frac{1}{e_{.j}} + \frac{1}{n_{.j} - e_{.j}} \right), \qquad (2.5)$$

which, he pointed out, does not necessarily follow a chi-square distribution. Under the null hypothesis,

$$E(X_a^2) = \sum_{j=1}^{t} V(X_{.j} \mid x_{i.}) \left(\frac{1}{e_{.j}} + \frac{1}{n_{.j} - e_{.j}} \right),$$
$$= \sum_{j=1}^{t} \sum_{i=1}^{b} V(X_{ij} \mid x_{i.}) \left(\frac{1}{e_{.j}} + \frac{1}{n_{.j} - e_{.j}} \right),$$

where the variances are given by (2.3). In general this expectation will depart from its nominal value of t-1. Armitage defines a corrected form of this statistic which has the proper expectation.

$$X_{corr.}^2 = (t-1)\, X_a^2 / E(X_a^2). \qquad (2.6)$$

A second test statistic suggested by Birch [1965] is based on the quadratic form,

$$X_q^2 = \underset{\sim}{d}' \, \underset{\sim}{A}^{-1} \, \underset{\sim}{d}, \qquad (2.7)$$

where the vector $\underset{\sim}{d}$ is formed by the deviations, $X_{.j} - e_{.j}$, $j=1,2,\ldots t-1$, and the matrix $\underset{\sim}{A} = [a_{jk}]$ is their variance-covariance matrix,

$$a_{jj} = \sum_{i=1}^{b} V(X_{ij} \mid x_{i.}), \qquad j=1,2,\ldots, t-1,$$
$$a_{jk} = \sum_{i=1}^{b} C(X_{ij}, X_{ik} \mid , x_{i.}), \quad j \neq k = 1,2,\ldots, t-1.$$

The statistic X_q^2 may be taken to have an asymptotic chi-square distribution with t-1 degrees of freedom.

A third test statistic might also be derived from the general likelihood ratio test criterion for composite hypotheses. This has been derived and discussed by Goodman [1970].

It is of interest to consider the various special cases of $X^2_{corr.}$ and X^2_q.

(i) If b=1, these two statistics reduce to the usual chi-square test for a 2 x t table apart from n-1 for n. If, in addition, t=2, they reduce to the usual uncorrected chi-square.

(ii) If t=2, these two statistics reduce to Cochran's [1954] for combining 2 x 2 tables apart from n-1 for n. If a continuity correction is introduced these statistics are equivalent to that suggested by Mantel and Haenszel [1959] and Cox [1966].

(iii) If t=2 and all n_{ij}=1. These statistics each reduce to McNemar's test (Mosteller [1952]) for matched proportions apart from the 1/2 correction.

(iv) If n_{ij}=1. These statistics each reduce to Cochran's [1950] Q statistic for testing several matched proportions.

3. Tests of the Model

The tests of the previous section are based on the assumption of "no interaction" in the 2 x b x t table. We consider now tests of that assumption. The more general model assumes

$$P_{ij} = \frac{\exp(\mu + \beta_i + \tau_j + \gamma_{ij})}{1 + \exp(\mu + \beta_i + \tau_j + \gamma_{ij})}, \qquad \begin{matrix} i=1,2,\ldots b \\ j=1,2,\ldots t \end{matrix}$$

where $\Sigma_j \gamma_{ij} = \Sigma_j \gamma_{ij} = 0$.

The null hypothesis for the model is H_1: $\gamma_{ij}=0$, for all i and j against H_2: $\gamma_{ij} \neq 0$ for some i and j. Note that H_1 here is the same H_1 as in Section 2. The likelihood for this model can easily be shown to be a function of the minimal set of sufficient statistics: $x_{..}$, $x_{i.}$, $x_{.j}$, and x_{ij}, which are related to the parameters μ, τ_i, β_j and γ_{ij}, respectively for all i and j. Thus inferences about the γ's should be based on the conditional distribution of the X_{ij}'s given the other components of the sufficient set of statistics. Under H_1, this is

$$f(X_{ij} \mid x_{..}, x_{i.}, x_{ij}; \gamma_{ij}=0) = \frac{\prod_{i,j} \binom{n_{ij}}{X_{ij}}}{\sum_{\Omega} \prod_{i,j} \binom{n_{ij}}{X_{ij}}}, \qquad (3.1)$$

where $\Omega = \{ X_{ij}: X_{..} = x_{..}, X_{i.} = x_{i.}, X_{.j} = x_{.j} \}$ for all i and j. An exact test of H_1 may be constructed by defining the appropriate rejection region. However this is a very onerous computing chore for large numbers.

Using a device of Zelen [1971], we divide the numerator and denominator of (3.1) by $\prod_{i,j} \binom{n_{ij}}{x_{i.}}$. Then the resulting t-variate hypergeometric distributions may be approximated, for large n's, by multivariate normal densities. Then we have (see Gart [1972] and Zelen [1971] for the details), that (3.1) may be approximated by exp (-Q/2) where,

$$Q = \sum_{i=1}^{b} \left(\frac{n_{i.}-1}{n_{i.}} \right) X_i^2 - X_q^2, \qquad (3.2)$$

where X_i^2, is the usual homogeneity chi-square for the 2 x t contingency table of the ith row. Q is suggested as a test statistic of H_1. Since

X^2_q is well approximated by $X^2_{corr.}$ we may also use

$$Q' = \sum_{i=1}^{b} \left(\frac{n_{i.} - 1}{n_{i.}} \right) X_i^2 - X^2_{corr.} \qquad (3.3)$$

as a test statistic. It has been shown that Q has an asymptotic chi-square distribution with (b-1) x (t-1) degrees of freedom. Equation (3.2) is the analogue in the untransformed space of Goodman's [1964] logit analysis statistic (his equation 2.2.8).

The statistics Q and Q' are very easy to compute in that they depend on the individual homogeneity chi-squares for the separate 2 x t tables and the adjusted, combined test for treatments. However, they suffer in comparison with other interaction tests in that they are not invariant with respect to the interchange of row and column labelling. However, the labelling difference does not appear to have an important effect on the size of the test statistic in any of the several tables I have analyzed so far by this method.

4. Applications of the Methods

In order to illustrate these methods we consider the data of Innes et al [1969] on pulmonary tumors among mice fed pesticides over an eighteen month experimental period. Each of the five control groups consisted of the two sexes of each of two strains of mice. The data and analysis are given in Table 1.

Table 1

Comparison of Five Control Groups (Innes et al [1969]) Adjusted for Four Subgroups, Frequency of Pulmonary Tumors by X^2_a, $X^2_{corr.}$, Q, and Q'.

Subgroup (Blocks)		Control Groups I	II	III	IV	V	$x_{i.}$ $n_{i.}$	2 x 5 Tables X^2_i	d.f.	$\left(\frac{n_{i.}-1}{n_{i.}}\right) X^2_i$
X Males	x_{1j}	2	2	0	1	0	5	4.12	4	4.069
	n_{1j}	17	15	14	17	16	79			
X Females	x_{2j}	1	1	1	0	0	3	1.90	4	1.877
	n_{2j}	18	18	18	17	16	87			
Y Males	x_{3j}	2	3	0	3	2	10	3.38	4	3.337
	n_{3j}	18	18	18	18	18	90			
Y Females	x_{4j}	0	0	0	2	1	3	6.12	4	6.046
	n_{4j}	17	15	18	15	17	82			
Marginal	$x_{.j}$	5	6	1	6	3	21	Total	16	15.329
Totals	$n_{.j}$	70	66	68	67	67	338	$X^2_q(X^2_{corr.})$	4	4.796(4.792)
								Q(Q')	12	10.533(10.537)

These results indicate no interaction in the 2 x 4 x 5 table and so significant "main effect" among the control groups.

On this basis, the fine control groups were combined to yield the basic control group given in Table 2 for comparison to a set of experimental mice fed Avadex.

Table 2

Analysis of Four Subgroups of Mice for Enhanced Frequency of Pulmonary Tumors Fed the Fungicide Avadex (Innes et al [1969])

Subgroups		Treated	Control	Totals	X_i^2	d.f.	$\left(\frac{n_{i.}-1}{n_{i.}}\right)X_i^2$
X Males	With Tumor	4	5	9	5.41	1	5.35
	Without Tumor	12	74	86			
	Totals	16	79	95			
X Females	With Tumor	2	3	5	2.40	1	2.37
	Without Tumor	14	84	98			
	Totals	16	87	103			
Y Males	With Tumor	4	10	14	1.64	1	1.63
	Without Tumor	14	80	94			
	Totals	18	90	108			
Y Females	With Tumor	1	3	4	0.29	1	0.29
	Without Tumor	14	79	93			
	Totals	15	82	97			
					Totals	4	9.64
					$X_q^2 (X_{corr.}^2)$	1	8.29(8.29)
					Q (Q')	3	1.35 (1.35)

Here the small value of Q (identical with Q' in 2 x 2 x b **tables)** indicates no significant interaction, while the significantly large value of $X_q^2 (\equiv X_{corr.}^2)$ indicates a great increase in pulmonary tumor incidence in the treated group compared to the control. Point and interval estimation for these examples are given in Gart [1971]. Other examples are analyzed by these methods in Gart [1970, 1972].

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time series
infection in man
null responses
reliability
maximum likelihood
ionospheric forecasting
disease severity index
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least squares
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industrial learning
clustering techniques
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